On the economics of harvesting age-structured fish populations*

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November 2007

Abstract

A generic discrete time age-structured population model is integrated with fishery economics to derive analytical results for maximizing present value returns net of effort cost. The simplest case is obtained by assuming two age classes. Given knife-edge selectivity and no harvesting cost, the optimal steady state is a unique (local) saddle point. When fishing gear is nonselective, optimal harvesting may converge toward a stationary cycle that represents pulse fishing. Optimal steady states differ from those obtained by the aggregate biomass models. This implies that optimal extinction results depend on age structured information. Given a low rate of interest and knife-edge selectivity, the optimal harvesting is shown to converge toward a unique (local) saddle point independently of the number of age classes.

Keywords: Fishery economics, optimal harvesting, age-structured populations

1 Introduction

Optimal harvesting of biologically renewable populations (such as fish and trees) is among the classical problems of resource economics. One line of research that dates back to Baranov (1918) aims to utilize information on the population age-structure in searching desirable levels of harvesting effort, or as is possible in some cases, the age class to which the harvesting activity should be targeted. The present study introduces an analytically solvable model for the optimal harvest of age-structured fish populations and suggests that it should be possible to proceed towards an analytical understanding comparable to that achieved for models describing the harvestable resource by a single variable for population biomass.

During the last forty years or more, mathematicians have analyzed these questions under continuous age and time structure by applying the Lotka-McKendrik model and its nonlinear generalizations. This setup leads to partial differential equations and calls for extended versions of optimal control theory (see e.g. Brokate 1985 or Hritonenko and Yatsenko 2007).

In fishery economics, the problem is approached by age-structured models specified in discrete time. Hannesson (1975) employs the classical Beverton-Holt (1957) multicohort model and shows numerically that for North Atlantic cod optimal harvesting takes the form of pulse fishing. In studies well known in population ecology (Caswell 2001), Horwood and Whittle (1986) and Horwood (1987) study the economics of harvesting age-structured populations applying a specific linearization technique in numerical computation. Recently, Stage (2006) applies an age-structured model to Namibian linefishing. Many papers have applied age-structured models to various kinds of policy analysis without attempting to solve the generic optimization problem (e.g. Sumaila 1997, Steinshamn 1998 and Björndal et al. 2004, Moxnes 2005).

In addition to fishery economics, the discrete time model has been used in studies concerning various management problems in fishery ecology. A seminal paper is Walters (1969) that applies numerical dynamic programming. Much of this literature is surveyed in Getz and Haight (1989) and Quinn and Deriso (1999). With closer inspection, it becomes evident that the studies in fishery ecology (excluding Walters 1969) solve the model under more or less *ad hoc* types of restrictions, such as requiring harvest (or effort) to be constant over time.

The discrete time approach for the age-structured fishery problem is well-grounded due to the facts that both reproduction and fishing activities may have clear seasonal characteristics. In addition, age-structured fish population models and data are normally given in a discrete time framework and these models have a long history and numerous management applications in population ecology (Leslie 1945, Caswell 2001).

Economic research in this field has been almost exclusively based on numerical computation. The reason for this has been the view that the age-structured fishery problem is analytically intractable (e.g. Wilen 1985, Clark 1985, 1990, 2006)[1]. Compared to the biomass framework commonly applied

in fishery economics (Gordon 1954, Plourde 1970), the complication follows because of many state variables represented by age classes. In addition, some age-structured specifications commonly applied in fishery ecology and economics (such as the Beverton-Holt 1957 formulation) may not, as such, represent the most fruitful basis for analytical work.

This study formulates the age-structured optimization model as a non-linear programming problem that can be analyzed by standard methods. In addition to the analytical approach, numerical computation is used to illustrate the results and to shed light on some questions that are beyond explicit solutions. The population model is in line with generic models applied in fishery ecology (Walters and Martell 2004) and it can be viewed as a direct generalization of the traditional "lumped parameter" model applied in fishery economics.

The general multiple age-class harvesting problem is complex, but there are no obstacles to obtain the necessary optimality conditions using the Kuhn-Tucher theorem. A fruitful setup for a theoretical analysis is obtained by assuming only two age classes. Given knife-edge fishing technology, harvesting affects only the older age class. In the absence of harvesting cost and under nonlinear utility, the interior steady state is a unique (local) saddle point equilibrium with clear comparative statics properties.

Under linear utility, the optimal solutions are found explicitly for initial states not too far from the steady state. It is shown that the constant escapement policy does not represent optimal solution, similarly to the biomass approach. Under nonselective fishing gear, it is proved that pulse fishing becomes the optimal solution. The intuition is that under certain conditions it is possible to avoid catching fish that are too small (i.e. to prevent "growth overfishing") by harvesting the population cyclically and only at periods when the proportion of older age class is high.

Since the age-structured model can be viewed as a generalization of the biomass model, it is possible to compare the optimal steady states of these models. The steady states coincide only under zero discounting. Under knife edge-selectivity, the biomass model yields larger steady state population, while the reverse may hold when fishing gear is nonselective. These results imply that the "optimal extinction" outcomes that have been extensively studied in resource economics (e.g. Olson and Roy 1996, 2000) are dependent on whether the resource is viewed as an aggregate biomass or as an age-structured system.

Assuming knife-edge selectivity and that only the oldest age class is harvested enables the steady state equation to be obtained without limiting the number of age classes. It is possible to prove the uniqueness of the steady state and that it is a saddle point equilibrium under the low rate of interest assumption. Numerical computation suggests that the general model version exhibits similar smooth harvesting vs. pulse fishing features than does the simplified model with two age classes.

The literature applying the discrete time age-structured model is almost entirely based on numerical computation. The contribution of this paper is in showing that there are no obstacles to study the generic age-structured optimal harvesting problem analytically. The specific results presented are new in

fisheries economic literature. Analytical work in this area helps to fulfil the cap between fishery ecology and economics literature (cf. Hilborn and Walters 2001, p. 472).[2]

The paper is organized as follows. Section two presents the model specification and the necessary optimality conditions. Section three considers the specification with two age classes and presents results for steady states, their stability and pulse fishing solutions. Section four develops the steady state equations for a model version with unlimited number of age classes and shows the saddle point feature of this equilibrium. The final section further discusses the discrete time age-structured model.

2 The age-structured problem and optimality conditions

Let x_{st}, $s = 1, ..., n$, $t = 0, 1, ...$ denote the number of fish in age class s at the beginning of period t. The number of eggs x_{0t} is given by

$$x_{0t} = \sum_{s=1}^{n} \gamma_s x_{st}. \tag{1}$$

where the constants $\gamma_s \geq 0$, $s = 1, ..., n$ are fecundity parameters. The recruits or the age class 1 fish are a function of the number of eggs, i.e.

$$x_{1,t+1} = \varphi(x_{0t}), \tag{2}$$

where φ is a recruitment function. Assume $\varphi(0) = 0$, $\varphi'(0) \leq 1$, $\varphi' \to 0$ when $x_0 \to \widetilde{x}_0$, and that φ is strictly concave when $x_0 \in [0, \widetilde{x}_0]$. As specific examples, this study applies the Beverton-Holt (1957) and Richer (1954) recruitment function given as

$$\begin{aligned}
\varphi_{BH}(x_{0t}) &= \beta_1 x_{0t}/(1 + \beta_2 x_{0t}), &\text{(3a)} \\
\varphi_R(x_{0t}) &= \beta_1 x_{0t} e^{-\beta_2 x_{0t}}. &\text{(3b)}
\end{aligned}$$

The Beverton-Holt (1957) recruitment function (3a) is strictly concave, while the Richer (1954) specification (3b) is concave for $x_0 \in [0, 2/\beta_2]$ and convex for $x_0 \in [2/\beta_2, \infty)$. Both are concave when $\varphi' > 0$. Denote the number of fish harvested from age class s at the end of periods by h_{st}, $s = 1, ..., n$, $t = 0, 1, ...$ The development of the age classes $s = 2, ..., n$ are given by

$$\begin{aligned}
x_{s+1,t+1} &= \alpha_s x_{st} - h_{st}, \ s = 1, ..., n-2, \ t = 0, 1, ..., &\text{(4a)} \\
x_{n,t+1} &= \alpha_{n-1} x_{n-1,t} - h_{n-1,t} + \alpha_n x_{nt} - h_{nt}, \ t = 0, 1, ..., &\text{(4b)}
\end{aligned}$$

where $0 < \alpha_s \leq 1$, $s = 1, ..., n$ are survival parameters. Equation (4a) for the oldest age class shows that fish remain in this age class if they have survived natural mortality and fishing. The number of fish harvested from each age class are specified as

$$h_{st} = \alpha_s x_{st} q_s(E_t), \ \ s = 1, ..., n, \ t = 0, 1, ..., \tag{5}$$

where the age class specific functions $q_s(E_t)$, $s = 1, ..., n$, $t = 0, 1...$ give fishing mortality. Assume that the functions q_s, $s = 1, ..., n$ are twice and continuously differentiable, nondecreasing, concave and $q_s(0) = 0$, $q_s(E) \to \widehat{q}_s$ as $E \to \widehat{E}_s$, where $0 \leq \widehat{q}_s \leq 1$ and $\widehat{E}_s > 0$ for $s = 1, ..., n$. One example that satisfies these assumptions is

$$h_{st} = \alpha_s x_{st} \widehat{q}_s (1 - e^{-\sigma_s E_t}), \ \ s = 1, ..., n, \ t = 0, 1, ..., \tag{6}$$

where σ_s, $s = 1, ..., n$ are nonnegative constants.

Given the weight of fish of age class s is $\phi_s \geq 0$, $s = 1, ..., n$ the total yield H_t is

$$H_t = \sum_{s=1}^{n} \phi_s h_{st}. \tag{7}$$

Obviously effort is restricted to be nonnegative, i.e. $E_t \geq 0$. In addition, there are restrictions to the lower bound number of fish in each age class. When $\widehat{q}_s = 1$, $s = 1, ..., n$ the lower bound restrictions become $x_s \geq 0$, $s = 1, ...n$. However, under knife edge selectivity, for example $q_s(E) = \widehat{q}_s = 0$ for some $s = 1, ..., j$ where $j < n$. More generally, when the maximum fishing mortality is below one, it is not possible to harvest the given age class to zero. To take these restrictions into account, the lower bounds are written as (cf. equations 2, 4a,b).

$$x_{1t} \geq 0, \tag{8a}$$
$$x_{s+1,t+1} \geq \alpha_s x_{st}(1 - \widehat{q}_s), \ \ s = 1, ..., n - 2, \tag{8b}$$
$$x_{n,t+1} \geq \alpha_{n-1} x_{n-1,t}(1 - \widehat{q}_{n-1}) + \alpha_n x_{nt}(1 - \widehat{q}_n). \tag{8c}$$

Let U denote an increasing and concave function for the utility of total yield, and C an increasing and convex function for fishing effort. A special case of the utility function is $U(H) = pH$, where p is a market price of fish. It is straightforward to specify an extension where the market price depends on the size (or age) of fish. Given $b = 1/(1+r)$ is the discount factor and r the rate of discount, the objective function to be maximized can be given as

$$\max_{\{E_t, \ x_{st}, \ s=1,...,n, \ t=0,1,...\}} V(\mathbf{x}_0) = \sum_{t=0}^{\infty} [U(H_t) - C(E_t)] b^t. \tag{9}$$

The optimization problem is now defined by (9) and by the constraints 2, 4a,b and 8a-c, where H_t, x_{0t}, and h_{st}, $s = 1, ..., n$ are given by (1), (5) and (7). In this study, this problem will be studied both analytically and numerically. All functions are continuous with continuous first and second order derivatives and it is possible to develop necessary conditions by applying the Kuhn-Tucker theorem[3]. These conditions will be used to obtain analytical results. Beyond this, the conditions are used by applying specific functions and parameter values

to obtain additional understanding numerically. Independently of this, the optimal solutions will be computed by applying gradient-based numerical solution methods. For these purposes, this study employes Knitro large scale optimization software (Byrd et al. 1999, 2006) that includes state-of-the-art interior (or barrier) and active set methods. By applying numerical methods it is possible to illustrate the analytical results and, in addition, their application enables an examination of the optimal transition dynamics and higher dimensional cases that are analytically untractable but interesting from the point of view of fisheries itself.

Let λ_{st}, $s = 1, ..., n$, $t = 0, 1, ...$ denote the Lagrange multipliers for the constraints (2), (4a,b). The Lagrangian function and the optimality conditions are

$$L = \sum_{t=0}^{\infty} b^t \{ U \left[\sum_{s=1}^{n} \phi_s \alpha_s x_{st} q_s(E_t) \right] - C(E_t) + \tag{10}$$
$$\lambda_{1t} \left[\varphi(\sum_{s=1}^{n} \gamma_s x_{st}) - x_{1,t+1} \right] + \sum_{s=1}^{n-2} \lambda_{s+1,t} \{ \alpha_s x_{st}[1 - q_s(E_t)]$$
$$- x_{s+1,t+1} \} + \lambda_{nt} \{ \alpha_{n-1} x_{n-1,t}[1 - q_{n-1}(E_t)] +$$
$$\alpha_n x_{nt}[1 - q_n(E_t)] - x_{n,t+1} \} \},$$

$$b^{-t} \frac{\partial L}{\partial E_t} = U' \left[\sum_{s=1}^{n} \phi_s \alpha_s x_{st} q_s(E_t) \right] \sum_{s=1}^{n} \phi_s \alpha_s x_{st} q_s'(E_t) - C'(E_t) - \tag{11}$$
$$\sum_{s=1}^{n-1} \lambda_{s+1,t} \alpha_s x_{st} q_s'(E_t) - \lambda_n \alpha_n x_{nt} q_n'(E_t) \leq 0, \ t = 0, 1, ...,$$
$$b^{-t} \frac{\partial L}{\partial E_t} E_t = 0, E_t \geq 0, \ t = 0, 1, ..., \tag{12}$$

$$b^{-t} \frac{\partial L}{\partial x_{1,t+1}} = bU' \left[\sum_{s=1}^{n} \phi_s \alpha_s x_{s,t+1} q_s(E_{t+1}) \right] \phi_1 \alpha_1 q_1(E_{t+1}) - \lambda_{1t} +$$
$$b\lambda_{1,t+1} \varphi'(\sum_{s=1}^{n} \gamma_s x_{s,t+1}) \gamma_1 + b\lambda_{2,t+1} \alpha_1 [1 - q_1(E_{t+1})] \leq 0, \tag{13}$$
$$b^{-t} \frac{\partial L}{\partial x_{1,t+1}} x_{1,t+1} = 0, \ x_{1,t+1} \geq 0, \ t = 0, 1, ..., \tag{14}$$

$$b^{-t} \frac{\partial L}{\partial x_{s+1,t+1}} = bU' \left[\sum_{s=1}^{n} \phi_s \alpha_s x_{s,t+1} q_s(E_{t+1}) \right] \phi_{s+1} \alpha_{s+1} q_{s+1}(E_{t+1}) +$$
$$b\lambda_{1,t+1} \varphi'(\sum_{s=1}^{n} \gamma_s x_{s,t+1}) \gamma_{s+1} - \lambda_{s+1,t} +$$
$$b\lambda_{s+2,t+1} \alpha_{s+1} [1 - q_{s+1}(E_{t+1})] \leq 0, \ s = 1, ..., n - 2, \ t = 0, 1, ..., \tag{15}$$

$$b^{-t} \frac{\partial L}{\partial x_{s+1,t+1}} [x_{s+1,t+1} - \alpha_s x_{st}(1 - \widehat{q}_s)] = 0, \tag{16a}$$
$$x_{s+1,t+1} - \alpha_s x_{st}(1 - \widehat{q}_s) \geq 0, \quad s = 1, ..., n - 2, \ t = 0, 1, ..., \tag{16b}$$

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$$b^{-t}\frac{\partial L}{\partial x_{n,t+1}} = bU'\left[\sum_{s=1}^{n}\phi_s\alpha_s x_{s,t+1}q_s(E_{t+1})\right]\phi_n\alpha_n q_n(E_{t+1})+ \qquad (17)$$

$$b\lambda_{1,t+1}\varphi'(\textstyle\sum_{s=1}^{n}\gamma_s x_{s,t+1})\gamma_n - \lambda_{nt}+$$

$$b\lambda_{n,t+1}\alpha_n[1-q_n(E_{t+1})] \le 0, \ t=0,1,...,$$

$$b^{-t}\frac{\partial L}{\partial x_{n,t+1}}[x_{n,t+1}-\alpha_{n-1}x_{n-1,t}(1-\widehat{q}_{n-1})-\alpha_n x_{nt}(1-\widehat{q}_n)]=0, \qquad (18a)$$

$$x_{n,t+1}-\alpha_{n-1}x_{n-1,t}(1-\widehat{q}_{n-1})-\alpha_n x_{nt}(1-\widehat{q}_n)\ge 0, \ t=0,1,.... \qquad (18b)$$

Next the optimal solutions are studied by proceeding from simple cases toward the more complex any number of age classes specifications.

3 Simplest case with two age classes

3.1 Steady state analysis
Given $n=2$ conditions (11), (13) and (17) take the form:

$$U'(H_t)[\phi_1\alpha_1 x_{1t}q_1'(E_t)+\phi_2\alpha_2 x_{2t}q_2'(E_t)]-C'(E_t)$$

$$-\lambda_{2,t}[\alpha_1 x_{1t}q_1'(E_t)+\alpha_2 x_{2t}q_2'(E_t)]\le 0, \ t=0,1,..., \qquad (19)$$

$$bU'(H_{t+1})\phi_1\alpha_1 q_1(E_{t+1})-\lambda_{1t}+b\lambda_{1,t+1}\varphi'(\gamma_1 x_{1,t+1}+$$

$$\gamma_2 x_{2,t+1})\gamma_1+b\lambda_{2,t+1}\alpha_1[1-q_1(E_{t+1})]\le 0, \qquad (20)$$

$$bU'(H_{t+1})\phi_2\alpha_2 q_2(E_{t+1})+b\lambda_{1,t+1}\varphi'(\gamma_1 x_{1,t+1}+\gamma_2 x_{2,t+1})\gamma_2-$$

$$\lambda_{2t}+b\lambda_{2,t+1}\alpha_2[1-q_2(E_{t+1})]\le 0, \ t=0,1,.... \qquad (21)$$

Assume a steady state where all the variables are constant over time, implying that the time subscripts can be cancelled. In addition, assume an interior solution where conditions (19)-(21) hold as equalities. Lagrangian multipliers λ_1 and λ_2 can be solved by (19) and (20) and eliminated from (21), which for interpretation purposes can be written in the form

$$\frac{\alpha_1 x_1 q_1'+\alpha_2 x_2 q_2'}{\partial U/\partial E}\left[\partial U/\partial x_2+\frac{\varphi'\gamma_2\partial U/\partial x_1}{(1+r)(1-\frac{\varphi'\gamma_1}{1+r})}\right]+\frac{\varphi'\gamma_2\alpha_1(1-q_1)}{(1+r)(1-\frac{\varphi'\gamma_1}{1+r})}+\alpha_2(1-q_2)-1=r.$$
$$((22))$$

Note that $(\alpha_1 x_{1t}q_1'+\alpha_2 x_{2t}q_2')/(\partial U/\partial E)=1/\lambda_2$. Equation (22) specifies an equality between the internal rate of return of a marginal unit of x_2 and the rate of interest. The fact that the rate of interest exists in both sides of the equation is a consequence of the delays in the the age class structure. The term

$\partial U/\partial x_2$ denotes the return an increase in x_2 causes in the harvest of x_2. A marginal increase in x_2 causes an increase in recruitment and thus an increase in x_1 equal to $\varphi'\gamma_2$. This increases the utility from harvest of x_1 by $\varphi'\gamma_2 \partial U/\partial x_1$. This return must be discounted over one period because of one period delay. In addition, the effect must be normalized by $1/[1 - \varphi'\gamma_1/(1+r)]$ because an increase in x_1 causes an increase in recruitment equal to $\varphi'\gamma_1$ units of x_1. Again, this must be discounted over one period due to one period delay. Both terms in the square brackets must be divided by λ_2 to obtain the *rate* of return of x_2. Next, the term $\varphi'\gamma_2\alpha_1(1-q_1)$ reflects the fact that an increase in x_2 increases x_1 via recruitment, and this in turn increases x_2 by the amount that is left after natural and fishing mortality. Again, these effects must be divided by $(1+r)[1-\varphi'\gamma_1/(1+r)]$ because of the time delay and the recruitment effects of x_1. Finally, the term $\alpha_2(1-q_2)$ gives the share of one unit of x_2 that is left after natural mortality and fishing. From these effect one must deduct unity because the age structured systems gives gross growth. Among other things, this steady state equation suggests that in age structured models the rate of interest plays a more complicated role than in models based on one single biomass variable.

It is illustrative to analyze the model when it is specified to certain cases instead of studying the most general version that includes all the effects simultaneously. The simplest possible case follows if juveniles do not belong to spawning stock and do not have direct economic value, i.e. $\phi_1 = 0$, $\gamma_1 = 0$. In addition, assume that fishing is costless $(C = 0)$ and that fishing gear has the knife-edge selectivity property in the sense that $q_1 = 0$. Let $\phi_2 = 1$. Under these assumptions, the first term in the LHS of (22) simplifies to $\alpha_2 q_2$ and equation (22) reduces to

$$\alpha_2 + \frac{\alpha_1 \varphi'(x_2\gamma_2)\gamma_2}{1+r} - 1 = r. \tag{23a}$$

Given $\alpha_2 < 1$, it must hold that $\varphi' > 0$ at equilibrium. Given φ is concave when $\varphi' > 0$, the steady state is unique. To give a simple intuition for (23a), note that $H = \alpha_1\varphi(x_2\gamma_2) + \alpha_2 x_2 - x_2$ yields the growth of x_2 that can be harvested under sustainability requirements. Thus, $\partial H/\partial x_2 = \alpha_1\varphi'\gamma_2 + \alpha_2 - 1$ is marginal growth. The term $\alpha_1\varphi'\gamma_2$ is the marginal effect on the next period level of x_2 that follows due to increased egg production and recruitment and after natural mortality of x_1. The term α_2 gives the share one unit of x_2 that is left after natural mortality.[4] From this, one must deduce 1 to obtain the marginal net effect on growth. At an optimal steady state, the marginal growth equals the rate of interest and $\alpha_1\varphi'\gamma_2$ must be discounted over one period.

Given a unique x_2 satisfying (23a), the other variables are determined by

$$x_1 = \varphi(\gamma_2 x_2), \tag{23b}$$

$$q_2(E) = \frac{\alpha_1\varphi(\gamma_2 x_2)}{\alpha_2 x_2} + 1 - 1/\alpha_2. \tag{23c}$$

$$H = \alpha_2 x_2 q_2(E). \tag{23d}$$

Implicit differentiation shows that increasing the rate of discount decreases the levels of x_2, x_1 and H, but increases the level of fishing mortality $q_2(E)$ and the level of effort E.

Equations (23a-d) specify an interior steady state in the sense that $x_2 > 0$. The steady state may exist only if $\alpha_2 + \alpha_1\varphi'(0)\gamma_2 - 1 > r$. However, the steady state is interior also in the sense that $q_2(E) \leq 1$ in (23c). This implies, for example, that when the condition $\alpha_2 + \alpha_1\varphi'(0)\gamma_2/(1+r) - 1 > r$ is violated the only remaining optimality candidate is not a solution that derives the population to "extinction".

As specified by restrictions (8b,c), it is possible that there exists some lowest attainable level of x_2 that becomes binding. This is most evident under knife-edge selectivity, where technological restrictions may prevent to harvest the population below some minimum level. For $\widehat{q}_2 = 1$, equation (23c) yields:

$$\mu(x_2) \equiv x_2 - \alpha_1\varphi(x_2\gamma_2) = 0, \tag{24a}$$

$$\partial\mu/\partial x_2 = 1 - \alpha_1\varphi'(x_2)\gamma_2. \tag{24b}$$

The equation $\mu(x_2) = 0$ has always at least one solution, i.e. $x_{2\infty} = 0$. If $1 - \alpha_1\varphi'(0)\gamma_2 < 0$, there exists another solution with some strictly positive x_2 because $\partial\mu/\partial x_2 \to 1$ when $x_2 \to \widetilde{x}_2$. This solution is the lowest attainable x_2 and denote it as $x_{2\,\min}$. It must hold that $\alpha_1\varphi'(x_{2\,\min})\gamma_2 < 1$. The question is whether $x_{2\,\min}$ may be lower than the level of x_2 that satisfies (23a). Recall that the value of x_2 that satisfies (23a) is higher the lower is the rate of interest. Assuming $r = 0$, equation (23a) reads as $1 - \alpha_2 - \alpha_1\varphi'(x_2)\gamma_2 = 0$. Letting α_2 approach zero, yields by strict convexity of φ, that $x_{2\,\min}$ is higher than the level of x_2 that solves (23a) under $r = 0$. Recall that $x_{2\,\min}$ must satisfy $\alpha_1\varphi'(x_{2\,\min})\gamma_2 < 1$. Since the solution to (23a) decreases with the rate of discount the outcome where $x_{2\,\min}$ becomes binding is more likely the higher is the rate of discount. These findings can be summarized as follows:

Proposition 1. Assume $n = 2$, $\phi_1 = \gamma_1 = q_1 = C = 0$, $\phi_2 = 1$, $\widehat{q}_2 = 1$ and that the recruitment function φ is concave when $\varphi' > 0$. The interior steady state is determined by (23a-d) and is unique. Total harvest and the number of fish in both age classes are decreasing and the fishing mortality and effort are increasing in the rate of interest. If $\frac{\alpha_1\varphi(\gamma_2 x_2)}{\alpha_2 x_2} + 1 - 1/\alpha_2 > 1$, where x_2 satisfies (23a), the steady state is independent of the rate of discount and is determined by (24a) and $x_1 = \varphi(x_2)$.

For further interpretation of the boundary steady state given by (24a), recall that this analysis considers the case of knife-edge selectivity ($q_1 = 0$, $q_2(E) \geq 0$). In addition, note that harvest occurs after spawning. Thus, recruits cannot be harvested and the share of recruits that survive as two period old fish spawn before they are all harvested. Under the assumption $1 - \alpha_1\varphi'(0)\gamma_2 < 0$, spawning of those fish that have just reach age class two is enough to maintain the population. The lowest attainable steady state level becomes the optimal steady state if the rate of discount is high enough. Increasing the rate of discount further does not affect the steady state. When the lowest attainable steady state

exceeds the MSY steady state, the optimal steady state is independent of the rate of discount and is determined by the knife edge fishing technology and biological factors only. Compared to biomass models, and especially compared to the interior steady state given by (22), the role of the rate of interest is now smaller.

Assuming Beverton and Holt (1957) recruitment function, $x_{2\min}$ is higher than the solution for (23a), e.g. if $r = 0$, $\beta_1 = 1$, $\beta_2 = 1/2$, $\alpha_1 = 9/10$, $\alpha_2 = 1/5$, and $\gamma_2 > 25/18$. However, if e.g. $\gamma_2 \leq 10/9$ (ceteris paribus), it follows that $x_{2\min} = 0$ and the steady state level for x_2 is determined by (23a).

3.2 Stability of steady states under nonlinear utility and no harvesting cost

To study the local dynamic properties of the interior steady state, write equation (4b) for the case $n = 2$ and $q_1 = 0$ as

$$x_{2,t+2} = \alpha_1 \varphi(x_{2t}) + \alpha_2 x_{2,t+1} - \alpha_2 x_{2,t+1} q_2(E_{t+1}), \qquad (25)$$

where it is assumed that $\gamma_2 = 1$ (without loosing generality). Since $C = q_1 = 0$, it is possible to take q_{2t} as an optimized variable instead of E_t. Write $q_{2t} = q_t$. In addition, assume $\alpha_1 = \alpha_2 = \alpha$. Since x_{1t} can be eliminated by $x_{1,t+1} = \varphi(x_{2t})$, write x_{2t} as x_t for simplicity. By equation (25) it is possible to obtain $q_{t+1} = q(x_t, x_{t+1}, x_{t+2})$, where $\partial q/\partial x_t = -\alpha\varphi'\partial q/\partial x_{t+2}$, $\partial q/\partial x_{t+1} = -\alpha(1 - q_{t+1})\partial q/\partial x_{t+2}$ and $\partial q/\partial x_{t+2} = -1/(\alpha x_{t+1})$. Next, it is possible to eliminate λ_{2t} and $\lambda_{2,t+1}$ by (19) and $\lambda_{1,t+1}$ by (20). After these steps, equation (21) takes the form

$$\theta = b^2\alpha U'[\alpha x_{t+3}q_{t+3}(x_{t+2}, x_{t+3}, x_{t+4})]\varphi'(x_{t+2})+$$
$$b\alpha U'[\alpha x_{t+2}q_{t+2}(x_{t+1}, x_{t+2}, x_{t+3})] - U'[\alpha x_{t+1}q_{t+1}(x_t, x_{t+1}, x_{t+2})] = 0. \quad (26)$$

This is a fourth order nonlinear difference equation. Write its characteristic equation as

$$w(u) = u^4 + w_4 u^3 + w_3 u^2 + w_2 u + w_1 = 0, \ where \qquad (27a)$$

$$w_1 = \frac{\partial\theta/\partial x_t}{\partial\theta/\partial x_{t+4}} = \frac{1}{b^2}, \qquad (27b)$$

$$w_2 = \frac{\partial\theta/\partial x_{t+1}}{\partial\theta/\partial x_{t+4}} = \frac{1}{b^2\varphi'} - \frac{\alpha}{b}, \qquad (27c)$$

$$w_3 = \frac{\partial\theta/\partial x_{t+2}}{\partial\theta/\partial x_{t+4}} = -\alpha\varphi' - \frac{U'\varphi''}{U''\varphi'} - \frac{\alpha}{b\varphi'} - \frac{1}{b^2\alpha\varphi'}, \qquad (27d)$$

$$w_4 = \frac{\partial\theta/\partial x_{t+3}}{\partial\theta/\partial x_{t+4}} = \frac{1}{b\varphi'} - \alpha. \qquad (27e)$$

Note first that $w(u) \to \infty$ as $u \to -\infty$ or $u \to \infty$. Since $w_4 > 0$ it follows that $w(0) > 0$. By the steady state condition (23a):

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$$w(1) = \frac{-U'\varphi''}{U''\varphi'} < 0,$$

$$w(-1) = \frac{-U'\varphi''}{U''\varphi'} + \frac{(b^2\alpha\varphi' - \alpha b - 1)(1 + \alpha - \varphi'\alpha)}{b^2\alpha\varphi'} < 0.$$

The sign of $w(-1) < 0$ follows from the concavity of U and φ and because at any interior steady state it must hold that $(b^2\alpha\varphi' - \alpha b - 1) < 0$ and $\alpha\varphi' < 1$. Thus, all the four roots of the characteristic equation are real and can be given as: $u_4 < -1 < u_3 < 0 < u_2 < 1 < u_1$. Thus the steady state is a local saddle point equilibrium.

Proposition 2. *Given the assumptions of Proposition 1 and a strictly concave utility function U, the interior steady state is a local saddle point.*

Examples of optimal solutions are shown in Figure 1. Applying the conditions (23a-d) the optimal steady state is $x_{1\infty} = 1.2127$, $x_2 = 2.355$, $H_\infty = 0.4991$ and applying the above derived formulas for characteristic roots imply $r_1 = -4.87$, $r_2 = -0.207$, $r_3 = 0.791$, $r_4 = 1.277$. Figure 1 is produced by the numerical optimization methods explained in section 2. It shows five examples of optimal solutions that converge toward the steady state. Excluding the origin, it is likely that the steady state is globally stable for optimal solutions.

3.3 Optimal solution under linear utility and no harvesting cost

The steady state defined by (23a-d) is independent of the utility function. However, it is evident that the same does not hold for optimal approach paths. To study the optimal trasitionary dynamics when the utility function is linear, assume $U' = 1$. For simplicity, maintain the assumptions $\gamma_1 = \phi_1 = q_1 = C = 0$, $\phi_2 = \gamma_2 = 1$. To find the initial states x_{10}, x_{20} that allow the optimal solution to reach the steady state in one period, assume $x_{11} = x_{1\infty}$ and $x_{21} = x_{2\infty}$, where the subscript ∞ refers to the steady state. Since $q_1 = 0$, it is again possible to take q_{2t} as the control variable and denote $q_{2t} = q_t$. By (2) and (4b) it must hold that

$$x_{11} = x_{1\infty} = \varphi(x_{20}), \tag{28}$$

$$x_{21} = x_{2\infty} = \alpha_1 x_{10} + \alpha_2 x_{20}(1 - q_0). \tag{29}$$

Thus, by (28) it holds that $x_{20} = x_{2\infty}$. In (29) $q_0 = 0$ implies $x_{10} = x_{2\infty}(1 - \alpha_2)/\alpha_1$ and $q_0 = 1$ that $x_{10} = x_{2\infty}/\alpha_1$. These are lower and higher bounds for x_{10}. Denote them by $\overline{x}_{10}$ and $\underline{x}_{10}$. (see Figure 2a). From (29) it follows that the fishing mortality is given as

$$q_0 = 1 - \frac{(x_{2\infty} - \alpha_1 x_{10})}{\alpha_2 x_{20}}. \tag{30}$$

This solution is optimal because it satisfies (19)-(21) as equalities with $\lambda_{2t} = 1$, $t = 0, 1, ...$

Consider next the possible initial states from which the steady state can be reached optimally with two steps. Since $x_{11} \in [x_{2\infty}(1 - \alpha_2)/\alpha_1, \; x_{2\infty}/\alpha_1]$ must hold after the first step, it follows by $x_{1,t+1} = \varphi(x_{2t})$ that the region for x_{20} is defined by

$$x_{2\infty}/\alpha_1 \leq \varphi(x_{20}) \leq x_{2\infty}(1 - \alpha_2)/\alpha_1. \tag{31}$$

Denote these lower and upper bounds by $\underline{x}_{20}$ and $\overline{x}_{20}$ accordingly (see Figure 1a). To reach the steady state with two steps it must also hold that $x_{21} = x_{2\infty}$, i.e. that $x_{2\infty} = \alpha_1 x_{10} + \alpha_2 x_{20}(1 - q_0)$. The highest possible level for x_{10} is obtained when $q_0 = 1$, and this level equals $x_{2\infty}/\alpha_1$. Finally, the left boundary of the region is found by setting $q_0 = 0$ implying a boundary $x_{20} = (x_{2\infty} - \alpha_1 x_{10})/\alpha_2$ (Figure 2a). By construction of these boundaries the initial fishing mortality that implies $x_{21} = x_{2\infty}$, is again given by (30). This solution satisfies conditions (19)-(21) since $x_{21} = x_{2\infty}$ and it is possible to set $\lambda_{2t} = 1$, $t = 0, 1, ...$. These findings can be summarized as:

Proposition 3. Given the assumptions of Proposition 1 and that $U' = 1$ and that an interior steady state exists, it is optimal to reach the steady state in one period if $x_{20} = x_{2\infty}$ and $x_{10} \in [x_{2\infty}(1 - \alpha_2)/\alpha_1, \; x_{2\infty}/\alpha_1]$. The optimal solution reaches the steady state in two periods if the initial state satisfies $x_{20} \geq (x_{2\infty} - \alpha_1 x_{10})/\alpha_2$, $x_{2\infty}/\alpha_1 \leq \varphi(x_{20}) \leq x_{2\infty}(1 - \alpha_2)/\alpha_1$ and $x_{20} \neq x_{2\infty}$.

Given the initial state that allows the optimal solution to reach the steady state in one or two steps, the solution is an example of constant escapement policy (see e.g. Spence 1973). Within the constant escapement policy, the population level after the harvest equals to its optimal after harvest steady state level. Such a policy is optimal for the biomass model under a linear objective and fishery production functions, and given that the beginning of period biomass level is not too low. The constant escapement policy means that the steady state is reached in one step. A similar policy may well be optimal for the age-structured model but with a more restrictive set of initial states. If escapement is interpreted to refer to the number of fish in age class 2, escapement is constant only for those initial states from which $x_{2\infty}$ can be reached optimally with one step. This is clearly not possible, if $x_{20} > \overline{x}_{20}$ (or $x_{10} > x_{2\infty}/\alpha_1$), for example. The reason why it takes more periods to reach the steady state is that high enough initial x_{20} implies high x_{11} and as an implication an excessively high x_{22}, i.e. $x_{22} > x_{2\infty}$. Compared to the biomass model, these complications are consequences of the age class structure.

An example demonstrating Proposition 3 is shown in Figure 2a,b. With the parameter values given, it follows that $x_{2\infty} \simeq 1.8644$, $x_{1\infty} \simeq 1.0680$, $\underline{x}_{10} \simeq 0.4661$, $\overline{x}_{10} \simeq 2.3304$, $\underline{x}_{20} \simeq 0.5729$, $\overline{x}_{20} \simeq 34.3617$. Figure 2a shows the sets of initial states that allow the steady state to become optimal in one or two periods. In addition, Figure 1a shows 5 examples where the steady state is reached in two periods. When the initial state is out of these sets it takes more than two periods to reach the steady state optimally. Examples of such solutions are shown in Figure 1b. It is likely that (excluding $x_{10} = x_{20} = 0$) the optimal steady state is globally stable for optimal solutions.

3.4 Steady state under growth overfishing

Assume next that $q_1 > 0$ but maintain the assumptions $\phi_1 = \gamma_1 = C = 0$, $\phi_2 = 1$. Thus, in this case fishing gear is nonselective in spite of the fact that age class one fish do not have commercial value and do not yet form part of the spawning stock. Fishery ecologists call such harvesting "growth overfishing", since fish are taken to be too small when harvested. Equation (22) for the optimal steady state obtains the form

$$\frac{q_2(E)\alpha_1 x_1 q_1'(E)}{x_2 q_2'(E)} + \frac{\alpha_1 \varphi'(\gamma_2 x_2)\gamma_2[1 - q_1(E)]}{1 + r} + \alpha_2 - 1 = r, \qquad (32)$$

where E is determined by $x_2 - \alpha_1\varphi(x_2)[1 - q_1(E)] - \alpha_2 x_2[1 - q_2(E)] = 0$. The assumption $q_1(E) = \tau q_2(E)$, where $0 \leq \tau \leq 1$ and the fact $x_1 = \varphi(x_2)$ simplifies (32) to

$$\Psi = \frac{\alpha_1 \tau q_2}{x_2}\left[\varphi - \frac{\varphi'\gamma_2 x_2}{1+r}\right] + \frac{\alpha_1 \varphi'(\gamma_2 x_2)\gamma_2}{1+r} + \alpha_2 - 1 = r, \qquad (33)$$

where $q_2 = (\alpha_1\varphi + x_2\alpha_2 - x_2)/(\alpha_1\varphi\tau + \alpha_2 x_2)$ and $\varphi - \varphi'\gamma_2 x_2/(1+r)$ is positive by the concavity of φ. Since the new expression $\div[\cdot]$ not existing in (23a) is positive, the steady state level of x_2 must be higher compared to the case $q_1 = 0$. Differentiation and the concavity of φ show that $\partial\Psi/\partial x_2 < 0$, implying that the optimal steady state is unique. In addition, since $\partial\tau q_2/\partial\tau > 0$, it follows that $\partial\Psi/\partial\tau > 0$, implying from the implicit function theorem that the steady state level of x_2 is increasing in τ. In addition, differentiation shows that $\partial q_2/\partial\tau < 0$ and $\partial H/\partial\tau = \partial(\alpha_2 x_2 q_2)/\partial\tau < 0$. These findings can be summarized as:

Proposition 4. Given that $\phi_1 = \gamma_1 = C = 0$, $\phi_2 = 1$ and $q_1(E) = \tau q_2(E)$, where $0 \leq \tau \leq 1$, the optimal steady state is unique and the steady state levels of x_1 and x_2 are increasing in τ, while the steady state levels of q_2 and H are decreasing in τ.

The stability of the steady state can be studied by applying similar steps as in section 3. However, the expressions will become rather tedious. Numerical computation of the characteristic roots shows that under strictly concave utility the steady state may still have the local saddle point property. However, setting $\tau = 1$, specifying $U = H^\sigma$ and letting $\sigma \to 1$ finally leads to an outcome with only one stable root. This follows, for example, if $U(H) = H^\sigma$, $\varphi(x_2) = x_2/(1 + 0.4x_2)$, $\alpha_1 = \alpha_2 = 0.8$, $b = \tau = 1$ and $0.9315 < \sigma < 1$. In these cases, it is optimal to reach the steady state only in hairline cases where the initial state x_{10}, x_{20} satisfies a specific functional relationship. The next task is to examine other possibilities for the long run equilibria.

3.5 Pulse fishing and cyclical equilibria under growth overfishing

Instead of a smooth equilibrium with constant harvest and population levels over time, another candidate for the long run equilibrium is the stationary cycle. This type of solution is known as pulse fishing (Walters 1969, Hannesson 1975) and intuitively it may become optimal due to the problem of growth overfishing.

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Proposition 4. Under the assumptions $\phi_1 = \gamma_1 = C = 0$, $U' = \phi_2 = \gamma_2 = 1$, $q_1 = q_2$, $\alpha_1 = \alpha_2$, $\alpha_2 \in (1/2, 1)$, $r = 0$ *and the Beverton-Holt recruitment function* $\varphi = \beta_1 x_{2t}/(1 + \beta_2 x_{2t})$ *with* $\beta_1 = 1$ *there exists an interior steady state and an optimal cyclical solution with cycle length equal to two periods.*

Proof: Appendix.

The analysis for the existence of the cyclical equilibrium leads to a ten equation nonlinear system. Due to its complexity, Proposition 4 shows the existence of the cyclical equilibrium under zero rate of interest and the Beverton-Holt specification. However, numerical analysis suggests that the cycle represents the optimal solution under a broad range of parameter values and, for example, under positive rate of interest. Simultaneously with the equilibrium cycle there exists a smooth steady state defined by equation (33). Depending, for example, on the convexity of the utility function, this equilibrium may have the saddle point characteristics or it may be totally unstable. Let us develop further intuition by studying a numerical example.

3.6 Numerical example of pulse fishing

To consider a numerical example assume: $\beta_1 = 1$, $\beta_2 = 0.4$, $\alpha = 0.8$, $q_1 = q_2 = 1$, $b = 1$. Using equation (33) yields the steady state: $x_1 \simeq 1.344$, $x_2 \simeq 2.906$, $q \simeq 0.145$, and $H \simeq 0.3378$. For computing the equilibrium cycle using equation (A11, see Appendix) yields $x_{22} \simeq 3.044$, $x_{21} \simeq 2.432$, $x_{11} \simeq 1.373$, $x_{12} \simeq 1.233$, $H_1 \simeq 0$, $H_2 \simeq 0.704$. In addition, the inequality (A12) obtains a value equal to -1.05. Note that in the cyclical equilibrium, the average per period yield, $H_2/2 \simeq 0.352$ is higher that the constant per period yield 0.3378 in the smooth equilibrium. Examples of solutions using numerical optimization methods (see section 2) are shown in Figure 3a,b,c. In Figure 3a, the solid lines show four solutions that all converge to the cyclical equilibrium that switches between the two circles. The solid line in Figure 3b shows the time path for one of the solutions. If the utility function is made strictly concave, the steady state with smooth harvest and population level becomes the long run equilibrium instead of the cycle. This is shown in Figure 3a by the two dashed lines that converge to the smooth equilibrium. These solutions have the same initial state as the associated solid lines but are based on $U = H^{0.5}$. An example of optimal yield over time is shown by the dashed line in Figure 3b. Assuming $U = H^{0.93}$ yields the solution that converges toward an interior limit cycle and that is given by the dotted line in Figure 3b.

Figure 3c shows the cyclical solution in effort, x_1, x_2 state space. The essential feature of the solution is that effort is zero at the periods when the level of x_1 is high and the level of x_2 is low. Consequently, the next period level of x_2 is high and x_1 is low and then the optimal effort is high. Thus, this pulse fishing strategy leads to a lower level of growth overfishing compared to solutions with smooth effort and fishing mortalities over time.

It is assumed in Proposition 4 that the rate of interest is zero. However, the cyclical equilibrium exists and also represents the optimal long run equilibrium under wide range of discounting. Figure 3d shows the lower and upper levels for the number of age class 2 fish as functions of the rates of interest.

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The smooth equilibrium with only one stable root is shown by dashed line for comparison.

3.7 Positive harvesting cost

If $C > 0$ the steady state equation becomes more complicated. Assuming $q_1 = \gamma_1 = 0$ and $\phi_2 = 1$, equation (22) can be written as

$$\Upsilon \equiv \frac{C'\alpha_2 q_2}{U'\alpha_2 x_2 q_2' - C'} + \alpha_2 + \frac{\alpha_1 \varphi'(x_2 \gamma_2)\gamma_2}{1 + r} - 1 = r. \tag{34}$$

Compared to (23a), the additional term in (34) reflects the marginal rate of return that follows since an increase in x_2 causes an increase in marginal utility of harvest without an increase in effort cost. Since the expression $C'\alpha_2 q_2/(U'\alpha_2 x_2 q_2' - C')$ in the LHS of (34) is positive, φ' must be lower and x_2^∞ must higher in order for (34) to hold compared to the case with zero effort cost. For studying the uniqueness of the steady state, note that $q(x_2) = 1 - [x_2 - \alpha_1\varphi(x_2)]/(\alpha_2 x_2)$ and differentiate the LHS of (34) with respect to x_2 :

$$\frac{\partial \Upsilon}{\partial x_2} = \left\{ \begin{array}{c} (C''q'\alpha_2 q + C'\alpha_2 q')(U''\alpha_2 x_2 q - C') - \\ C'\alpha_2 q[U''\alpha_2(x_2 q' + q)\alpha_2 x_2 + U'\alpha_2 - C''q'] \end{array} \right\} / (U'\alpha_2 q - C')^2.$$

Since $q' < 0$ and $x_2 q' + q$ may have any sign, the steady state may not be unique. However,. if e.g. $U = pH$, where p is the exogenous market price, it follows that $U'' = 0$ and $\partial \Upsilon/\partial x_2 < 0$ and that the steady state is unique.

The stability properties of the steady states can be studied by similar steps as in section 3.2. If the steady state is unique, one may expect saddle point stability if the possibility of stationary cycles can be ruled out. In the case of multiple steady states, some of the equilibria may be totally unstable.

3.8 Comparing the optimal steady states for the age-structured and biomass models

The surplus production model can be viewed as a simplification of the age-structured framework. This makes it possible to analyze how the age-structured information changes the optimal solutions from those that are based on the aggregate biomass information. The simplest possibility is to compare the steady states. For this purpose, assume $q_1 = \phi_1 = C = 0$ and $\gamma_1 > 0$, $\gamma_2 > 0$. In addition, assume that the lowest attainable level of x_2 is zero. Under these assumptions, any steady state must satisfy

$$x_1 = \varphi(\gamma_1 x_1 + \gamma_2 x_2), \tag{35}$$
$$x_2 = \alpha_1 x_1 + \alpha_2 x_2[1 - q_2(E)]. \tag{36}$$

Equation (35) can be used to derive x_1 as a function of x_2. Denote this as $x_1 = x_1(x_2)$, where $x_1' = \varphi'\gamma_2/(1 - \varphi'\gamma_1) > 0$. Equation (36) can be written as

$$H = \alpha_1\varphi[\gamma_1 x_1(x_2) + \gamma_2 x_2] + \alpha_2 x_2 - x_2, \tag{37}$$

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where $H[= \alpha_2 x_2 q_2(E)]$ is the equilibrium yield as a function of x_2. Since x_1 is not harvestable and its market value is zero ($q_1 = \phi_1 = 0$), it is natural to take x_2 to represent the biomass variable in the surplus production model. Thus, equation (37) can be considered to represent an equilibrium harvest as a function as biomass. Denote this relationship by $H = F(x_2)$. Given the harvesting cost is zero, the well known optimal steady state condition (e.g. Plourde 1970) for the surplus production model satisfies $F'(x_2) = r$ and $H = F(x_2)$. Differentiation and some cancellation yields

$$\frac{\partial H}{\partial x_2} = r \implies \frac{\alpha_1 \varphi'[\gamma_1 x_1(x_2) + \gamma_2 x_2]\gamma_2}{1 - \varphi'[\gamma_1 x_1(x_2) + \gamma_2 x_2]\gamma_1} + \alpha_2 - 1 = r. \tag{38}$$

From (22) the steady state equation for the age-structured model is

$$\frac{\alpha_1 \varphi'[\gamma_1 x_1(x_2) + \gamma_2 x_2]\gamma_2}{(1 + r)\{1 - b\varphi'[\gamma_1 x_1(x_2) + \gamma_2 x_2]\}\gamma_1} + \alpha_2 - 1 = r. \tag{39}$$

Differentiation shows that the term $\eta \equiv \alpha_1 \varphi' \gamma_2 / [(1 + r)(1 - b\varphi'\gamma_1)]$ is decreasing in x_2. Thus, the equilibrium for both the surplus production and the biomass models are unique. Next $\partial \eta / \partial r < 0$, implying that the level of x_2 defined by the equilibrium condition for the age-structured model is lower than the equilibrium defined by the surplus production model.

Assume next that $\phi_1 = \gamma_1 = C = 0$, $\phi_2 = 1$ and $q_1 = \tau q_2$. The steady state must satisfy $x_2 = \alpha_1 \varphi(x_2 \gamma_2)(1 - \tau q_2) + \alpha_2 x_2(1 - q_2)$. This yields $q_2 = [\alpha_1 \varphi(x_2 \gamma_2) + x_2(\alpha_2 - 1)] / [\alpha_1 \varphi(x_{2\gamma_2})\tau + \alpha_2 x_2]$. By using the fact $H = \alpha_2 x_2 q_2$ one obtains

$$H = F(x_2) = \frac{\alpha_2 x_2 \alpha_1 \varphi(x_2 \gamma_2) + \alpha_2 x_2^2(\alpha_2 - 1)}{\alpha_1 \varphi(x_2 \gamma_2)\tau + \alpha_2 x_2}. \tag{40}$$

Differentiation (40) and applying the expression for q_2 enables to write the condition $F'(x_2) = r$ to the form

$$\left\{ \frac{\alpha_1 \tau q_2}{x_2}[\varphi - \varphi' \gamma_2 x_2] + \alpha_1 \varphi' \gamma_2 + \alpha_2 - 1 \right\} \frac{\alpha_2 x_2}{\alpha_1 \varphi \tau + \alpha_2 x_2} = r. \tag{41}$$

Comparing (41) and the condition for the age structured specification (33) shows that there are two differences with the steady state conditions. In (33), the term $\alpha_1 \varphi' \gamma_2 (1 - \tau q_2)$ is divided by $1 + r$ and in (33) the term $\alpha_2 x_2 / [\alpha_1 \varphi(x_2)\tau + \alpha_2 x_2]$ does not exist. Note that if $r = 0$, the two conditions are equivalent. Assume $r > 0$ and $0 < \tau \leq 1$. Recall that the LHS of (33) is decreasing in x_2. Taking into account that $0 < \alpha_2 x_2 / [\alpha_1 \varphi(x_2)\tau + \alpha_2 x_2]$, the fact that $1 + r$ does not exist in (41) implies a positive effect on the level of x_2 that solves (41). However, since $\alpha_2 x_2 / [\alpha_1 \varphi(x_2)\tau + \alpha_2 x_2] < 1$, the effect of the other difference between the equations is reverse. Whether the steady state level of x_2 determined by condition (41) is lower or higher than the equilibrium level for the age-structured model depends, for example, on the level of τ.

A numerical comparison of these equilibria is presented in Figure 4a,b. Given nonselective gear ($\tau = 1$), the age structured model implies larger steady

state levels compared to the biomass model (Figure 4a). The situation is reverse under knife-edge selectivity ($\tau = 1$) (Figure 4b). Thus, the age-structured model implies a different steady state compared to the biomass model that is based on a single aggregate variable. This also implies that the extensively studied "optimal extinction" question (e.g. Olson and Roy 1996, 2000) is dependent on whether the age structured information is included in the analysis.

4 Equilibria with any number of age classes

4.1 Steady state analysis
With any number of age classes, the necessary conditions and the equations for steady states become complicated. However, it is still possible to develop steady state equations for some simplified cases. For this purpose, write the steady state for the age-structured system as

$$
\begin{align}
x_{s+1} &= \mu_s x_s, \ s = 1, ..., n-1 \text{ where} \tag{42a} \\
\mu_s &= \alpha_s(1 - q_s), \ s = 1, ..., n-2, \tag{42b} \\
\mu_{n-1} &= \alpha_{n-1}(1 - q_{n-1})/[1 - \alpha_n(1 - q_n)]. \tag{42c}
\end{align}
$$

Next, using (42a-c) it is possible to write all the age classes $s > 1$ as linear functions of the number of fish in age class 1:

$$
\begin{align}
x_{s\infty} &= \Phi_s x_1, \ s = 2, ..., n \text{ where} \\
\Phi_s &= \prod_{i=1}^{s-1} \mu_i, \ s = 2, ..., n.
\end{align}
$$

The equilibrium number of eggs equals $x_0 = x_1 \sum_{s=1}^{n} \gamma_s \Phi_s$, where $\Phi_1 \equiv 1$. Denote $\sum_{s=1}^{n} \gamma_s \Phi_s \equiv R$. Next, the number of fish in the age class 1 is specified as

$$
x_{1\infty} = \varphi(x_1 R). \tag{43}
$$

For the equilibrium level of x_1 to be strictly positive it is necessary that $1 - \varphi'(0)R < 0$.

To proceed for analyzing the rest of the steady state equations assume $q_s = \phi_s = 0, \ s = 1, ..., n-1, \ q_n > 0, \ C > 0$ and $\phi_n = 1$. Thus, harvesting technology has knife-edge characteristics and only fish in age class n and older belong to the spawning stock. This model specification extends the case studied in sections 3.1 and 3.2 and comes close to the delay difference model that is well known in fishery ecology (see e.g. Hilborn and Walters 2001). The only difference here is that all fish in age class n are of equal size, while in the population model analyzed by Hilborn and Walters (2001) the possibility that fish grow in age class n is included.

Applying (43) it is possible to obtain x_1 as a decreasing function of q_n:

$$
\frac{dx_1}{dq_n} = \frac{\varphi' x_1 \partial R/\partial q_n}{1 - \varphi' R} < 0,
$$

where $\partial R/\partial q_n = -\prod_{i=1}^{n-2}\frac{\alpha_{n-1}\alpha_n}{1-\alpha_n(1-q_n)} < 0$. Denote this function by $x_1 = x_1(q_n)$. The remaining optimality conditions are

$$U'\alpha_n x_n q'_n - C' - \lambda_n\alpha_n x_n q'_n = 0, \tag{44}$$

$$-\lambda_s + b\lambda_1\varphi'\gamma_s + b\lambda_{s+1}\alpha_s = 0, \ s = 1, ..., n-1, \tag{45}$$

$$bU'\alpha_n q_n + b\lambda_1\varphi'\gamma_n - \lambda_n + b\lambda_n\alpha_n(1-q_n) = 0. \tag{46}$$

Equations (45) can be solved recursively starting from $s = n - 1$ and proceeding toward $s = 1$. Each Lagrangian multiplier is then given as a function of λ_1 and λ_n and finally the equation for $s = 1$ defines λ_1 as a function of λ_n. Using this result, λ_1 can then be eliminated from (46). Dividing (46) by $b\lambda_n$ and solving (44) for λ_n yields:

$$\frac{U'\alpha_n^2 x_n q_n q'_n}{U'\alpha_n x_n q'_n - C'} + \frac{\varphi'\gamma_n b^{n-1}\prod_{j=1}^{n-1}\alpha_{n-j}}{1 - \varphi'\left\{\sum_{j=1}^{n-1}b^{n-1}\gamma_j\prod_{k=1}^{j-1}\alpha_{k-1}\right\}} + \alpha_n(1-q_n) = 1/b. \tag{47}$$

Equation (47) can be viewed to contain q_n as its single variable since all the state variables, as well as E in C', can be given as functions of q_n. If $C' = 0$, this equation has a unique solution since the first quotient in the LHS cancels with $-\alpha_n q_n$ and the remaining part is decreasing in x_1 and increasing with q_n.

4.2 Stability of the steady state

For analyzing the stability of the optimal steady state assume that $\gamma_s = 0$, $s = 1, ..., n-1$, $\gamma_n = 1$, $q_s = 0$, $s = 1, ..., n-1$. In addition, assume $C = 0$. It is now possible to take q_{nt} directly as the control variable. Denote it $q_{nt} \equiv q_t$ for simplicity. Given an interior solution, the conditions (11)-(18) take the form

$$U'(\alpha_n x_{nt}q_t)\alpha_n x_{nt} - \lambda_{nt}\alpha_n x_{nt} = 0, \tag{48}$$

$$-\lambda_{1t} + b\lambda_{2,t+1}\alpha_1 = 0, \tag{49}$$

$$-\lambda_{2t} + b\lambda_{3,t+1}\alpha_1 = 0, \tag{50}$$

$$\cdots,$$

$$\lambda_{n-1,t} + b\lambda_{n,t+1}\alpha_{n-1} = 0, \tag{51}$$

$$bU'(\alpha_n x_{n,t+1}q_{t+1})\alpha_n q_{t+1} + b\lambda_{1,t+1}\varphi'(x_{n,t+1}) - \lambda_{nt} + \alpha_n b\lambda_{n,t+1}(1-q_{t+1}) = 0. \tag{52}$$

Equations (49)-(51) yield $\lambda_{nt} = \lambda_{1,t-(n-1)}/(b^{n-1}\prod_{i=1}^{n-1}\alpha_i)$ and $\lambda_{1,t+1} = \lambda_{n,t+n}\sigma$, where $\sigma = b^{n-1}\prod_{i=1}^{n-1}\alpha_i$. In addition, the conditions $x_{1,t+1} = \varphi(x_{nt})$, $x_{2,t+1} = x_{1t}\alpha_1, ..., x_{n,t+1} = \alpha_{n-1}x_{n-1,t} + \alpha_n x_{nt}(1-q_t)$ yield

$$q_t = 1 - \frac{x_{n,t+1} - \sigma\varphi(x_{n,t-n+1})}{\alpha_n x_{nt}}.$$

Note from (1) that $\lambda_{nt} = U'(\alpha_n x_{nt} q_t)$. It is now possible to eliminate $\lambda_{1,t+1}$, λ_{nt} and q_t from (52) and obtain the following difference equation for x_{nt} :

$$
\begin{aligned}
\Omega \;=\;& b^n \varphi'(x_{t+2}) \sigma U' \left\{ \alpha_n x_{t+n+1} \left[1 - \frac{x_{t+n+2} - \sigma\varphi(x_{t+2})}{\alpha_n x_{t+n+1}} \right] \right\} \\
& - U' \left\{ \alpha_n x_{t+1} \left[1 - \frac{x_{t+2} - \sigma\varphi(x_{t-n+2})}{\alpha_n x_{t+1}} \right] \right\} \\
& + \alpha_n b U' \left\{ \alpha_n x_{t+2} \left[1 - \frac{x_{t+3} - \sigma\varphi(x_{t-n+3})}{\alpha_n x_{t+2}} \right] \right\},
\end{aligned}
$$

where $x_{nt} = x_t$ for simplicity. This is a nonlinear difference equation of order $2n$. To form its characteristic equation compute:

$$
\begin{aligned}
w_1 \;&=\; \frac{\partial\Omega/\partial x_{t-n+2}}{\partial\Omega/x_{t+n+2}} = \frac{1}{b^n}, \\[6pt]
w_2 \;&=\; \frac{\partial\Omega/\partial x_{t-n+3}}{\partial\Omega/x_{t+n+2}} = -\frac{\alpha_n}{b^{n-1}}, \\[6pt]
w_3 \;&=\; \frac{\partial\Omega/\partial x_{t+1}}{\partial\Omega/x_{t+n+2}} = \frac{\alpha_n}{b^n \varphi' \sigma}, \\[6pt]
w_4 \;&=\; \frac{\partial\Omega/\partial x_{t+2}}{\partial\Omega/x_{t+n+2}} = -\frac{\alpha_n^2}{b^{n-1}\varphi'\sigma} - \sigma\varphi' - \frac{1}{b^n \varphi'\sigma} - \frac{\varphi''U'}{\varphi'U''}, \\[6pt]
w_5 \;&=\; \frac{\partial\Omega/\partial x_{t+3}}{\partial\Omega/x_{t+n+2}} = \frac{\alpha_n}{b^{n-1}\varphi'\sigma}, \\[6pt]
w_6 \;&=\; \frac{\partial\Omega/\partial x_{t+n+1}}{\partial\Omega/x_{t+n+2}} = -\alpha_n.
\end{aligned}
$$

The characteristic polynomial can be written as

$$
\Phi(u) = u^{2n} + w_6 u^{2n-1} + w_5 u^{n+1} + w_4 u^n + w_3 u^{n-1} + w_2 u + w_1 = 0. \qquad (53)
$$

Let $r = 0$, i.e. $b = 1$. Direct substitution shows that if u is a root of (53) also $1/r$ is a root of (53). In addition, by direct substitution and the use of the steady state condition $b^{n-1}\varphi'\sigma + \alpha_n - 1/b = 0$ it follows that

$$
\Phi(1) = -\frac{U'\varphi''}{U''\varphi'} < 0,
$$

$$
\Phi(-1) = -\frac{U'\varphi''}{U''\varphi'} - \frac{4(\varphi'\sigma - 1)^2}{4\sigma} < 0, \quad \text{when } n \text{ is even}
$$

$$
\Phi(-1) = \frac{U'\varphi''}{U''\varphi'} + \frac{(\varphi'\sigma)^2 + 2\varphi'\sigma(\alpha_n + 1) + \alpha_n^2 + 2\alpha_n + 1}{\varphi'\sigma} > 0.
$$

19

The product of the roots equal 1 ($= w_1$ by Vieta's formula). The number of roots is $2n$, i.e. even. Since $\Phi(1) \neq 0$ and $\Phi(-1) \neq 0$ it follows that half of the roots lie inside the unit circle (centered at the origin) of the complex plane and half of the roots lie outside the unit circle (centered at the origin) of the complex plane. Thus, the steady state is a local saddle point when $r = 0$. Since Φ is continuously differentiable with respect to r, the saddle point property must hold for positive levels of r that are small enough. This proves the following:

Proposition 5. Given $q_s = \phi_s = 0$, $s = 1, ..., n-1$, $q_n > 0$, and $C = 0$ the steady state is a local saddle point when $r \geq 0$ is small enough.

The saddle point feature implies that the optimal solution converges toward the steady state equilibrium at least when the rate of interest is not too high and the initial state is not too far from the steady state. Note that this specification is a direct extension of the formulation studied in proposition 2, where under the assumption $n = 2$ it was possible to show that the saddle point feature holds independently of the rate of discount. Since the steady state is unique, it is likely that optimal solutions converge toward the equilibrium even for cases of large deviations from the steady state. Another question is the nature of optimal solutions when the rate of discount is not small. Both of these questions are next studied applying numerical optimization methods.

4.3 Numerical examples for populations with any number of age classes

Proposition 5 shows the local stability properties of the steady state under low rates of interest. Figure 5a shows the development of three optimal solutions over time. The solutions differ only because of their different initial age-class structures. All the solutions converge toward the same equilibrium. Figure 5b shows the same solutions in x_8, H state space. Both Figures demonstrate the nonmonotonicity of the optimal path. Figure 5b also suggests that there may be a linear boundary containing the optimal steady state such that the optimal spawning stock-yield combinations exist below or on this boundary. The solutions exist below the boundary if the number of fish in the age classes $1, ..., n-1$ are low, implying that it is optimal to keep the harvest at a low level even if the spawning stock and harvestable age class is near its long run optimal steady state level. Conversely, if the number of fish in age classes $s = 1, ..., n-1$ is higher, the optimal yield tends to be an approximately linear function of the spawning stock. These solutions reflect the fact that age classes $s = 1, ..., n-1$ contain valuable information on future harvesting possibilities and that information partly determines the optimal harvest at the present period.

Finally, Figure 6 depicts three optimal solutions for the general model without simplifications. There are eight age classes. The dotted line shows the pulse fishing solution where fishing gear is nonselective and young age classes do not have commercial value. This outcome is in line with Proposition 4. If the utility function is made more concave, the pulse fishing property becomes somewhat smoother but the solution may still have a limit cycle property (solid line). The solution given by the dashed line shows that the pulse fishing strategy completely disappears if the young classes are commercially valuable.

5 Conclusions

The aim of this paper has been to present analytical results and some numerical computation illustrations on optimal harvesting of age-structured fish populations. It is shown that the complexity of the multiple age class problem can be reduced by deriving results for model specification with only two age classes. It may be expected that the results obtained for this simplest case at least partially carry over to the specifications with higher number of age classes.[5] The paper takes some steps toward this direction by deriving a steady state equation and stability analysis for one important special case of the general model. In contrast to some opinions (Wilen 1985, Hilborn and Walters 2001, Clark 1985, 1990, 2006), the age structured model is found to be analytically tractable.

Some potential extensions are worth noting. The cyclical equilibrium is proved to exist given a zero rate of interest and a recruitment function by Beverton and Holt (1957). Numerical examples suggests that the cyclical equilibrium may be optimal under positive rates of interest and a Richer (1954) recruitment function. Thus, generalizations of the analytical result should be possible. Given knife-edge selectivity and no harvesting cost, the steady state for the two age class specification was shown to be a local saddle point independently of the rate of interest. The analogous result for the any number of age classes specification was proved under low rate of interest, but generalizations should again be possible. Finally, there should be no obstacles to study the empirically highly relevant case with stochastic recruitment and to derive some analytical results for the two age classes specification and numerical results for more complex age distributions.

Footnotes:

[1]Wilen (1985) writes: "If we are interested in real-world management problems, we are inevitably forced to disaggregate and to pick up the more complicated features of mixed aged populations. Unfortunately, these appear to be the most intractable analytically".

[2]In their extensively used book, Hilborn and Walters (2001) make a sharp distinction between analytically and numerically solvable models in fishery management. They write that one state variable biomass models are fruitful for elegant analytical solutions but too simple for people working in fisheries management. Models with explicit age structure are considered to be useful for practical management problems but beyond analytical methods.

[3]On the use of the Kuhn-Tucker theorem or the Lagrange method for solving discrete time dynamic optimization problems, see e.g. Mercenier and Michel (1994).

[4]Note that given $C = 0$, the term $-q_2\alpha_2$ cancels out with the direct effect an increase of x_2 has on the harvest of x_2.

[5]This is more or less the case in specific class of age structured models in forest economics (cf. Salo and Tahvonen 2002, 2003).

Appendix. Proof of Proposition 4.

Since $q_{1t} = q_{2t}$ it is possible to take q_{2t} as the optimized variable. Write $q_{2t} = q_t$ for simplicity. The conditions (19)-(21) take the form

$$\alpha x_{2t} - \lambda_{2t}\alpha(x_{1t} + x_{2t}) \leq 0, \tag{A1}$$

$$-\lambda_{1t} + b\lambda_{2,t+1}\alpha(1 - q_{t+1}) \leq 0, \tag{A2}$$

$$b[\alpha q_{t+1} + \lambda_{1,t+1}\varphi'(x_{2,t+1}) + \lambda_{2,t+1}\alpha(1 - q_{t+1})] - \lambda_{2t} \leq 0. \tag{A3}$$

The purpose is to first show that a two period cycle satisfies the necessary optimality conditions. For this end set $q_1 = 0$, $q_2 > 0$, x_{1i}, $x_{2i} > 0$, $i = 1, 2$ (Note that q_1 refers to period 1 fishing mortality etc.). This leads to the system

$$\alpha x_{2i} - \lambda_{2i}\alpha(x_{1i} + x_{2i}) \leq 0, \ i = 1, 2, \tag{A4}$$

$$-\lambda_{1i} + b\lambda_{2,i+1}\alpha(1 - q_{i+1}) = 0, \ i = 1, 2, \tag{A5}$$

$$b\alpha q_{i+1} + b\lambda_{1,i+1}\varphi'(x_{2,i+1}) + b\lambda_{2,i+1}\alpha(1 - q_{i+1}) - \lambda_{2i} = 0, \ i = 1, 2, \tag{A6}$$

$$x_{1i} = \varphi(x_{2,i+1}), \ i = 1, 2, \tag{A7}$$

$$x_{2i} = \alpha x_{1,i+1} - \alpha x_{1,i+1}q_{i+1} + \alpha x_{2,i+1} - \alpha x_{2,i+1}q_{i+1}, \ i = 1, 2, \tag{A8}$$

where, due to two period cycle, it is written that $q_3 = q_1$, $x_{13} = x_{11}$, etc. This is a 10 equations and variables system and the aim is to eliminate all variables excluding x_{22} and study the remaining equation (A6) for $i = 2$.

Eliminating x_{11}, x_{12}, x_{21} from (A7) and (A8) yields

$$x_{22}/\alpha - \varphi(x_{22}) - \alpha\{\varphi[x_{22}/\alpha - \varphi(x_{22})] + x_{22}\}(1 - q) = 0. \tag{A9}$$

This equation defines q as an decreasing function of x_{22}. Next λ_{11} and λ_{12} can be eliminated from (A6) using (A5). From (A4) it follows that $\lambda_{22} = x_{22}/(x_{21} + x_{22})$ and $\lambda_{21} \geq x_{21}/(x_{11} + x_{21})$. After eliminating λ_{11} from equation (A6) written for $i = 1$ one obtains an equation for λ_{21} which can be then be eliminated from (A6) written for $i = 2$. After these steps equation (A6) for $i = 2$ can be written as

$$\frac{x_{22}}{\varphi(x_{21}) + x_{22}} \left\{ b\alpha(1 - q) + \frac{[1 - b^2\alpha(1 - q)\varphi'(x_{21})][b^2\alpha\varphi'(x_{22}) - 1]}{b\alpha} \right\} + b\alpha q = 0, \tag{A10}$$

where $x_{21} = x_{22}/\alpha - \varphi(x_{22})$. Since (A9) defines q as a function of x_{22} equation (A5) includes x_{22} as its single variable. The task is to study the existence and uniqueness of solutions for equation (A10). Under the assumptions $\varphi = \beta_1 x_{22}/(1 + \beta x_2)$, $\beta_1 = 1$, and $b = 1$ (A10) becomes a fourth order polynomial

$$P(x_{22}) \equiv \beta_2^4(\alpha^2 - 1)x_{22}^4 + 4\beta_2^3(\alpha^2 - 1)x_{22}^3 + \beta_2^2(6\alpha^2 + 3\alpha - 5)x_{22}^2$$
$$+ 2\beta_2(\alpha^2 + \alpha - 1)x_{22} + \alpha(2\alpha - 1) = 0. \tag{A11}$$

The assumption that $\alpha > \frac{1}{2}$ implies that $\alpha(2\alpha - 1) > 0$ and $\beta_2^4(\alpha^2 - 1) < 0$. Thus $P(0) > 0$ and $P(x_{22}) \to -\infty$ as $x_{22} \to \infty$ showing the existence of at least one positive real root. Note that the coefficients for x_{22}^4 and x_{22}^3 are both negative. Thus, the sign of the coefficients cannot change three times ruling out the possibility of three positive real roots (cf. Descartes´ rule of signs). This shows that there exists only one positive real root for equation (A11).

The remaining necessary condition that must be satisfied is (A4) written for $i = 1, i.e.$ $\lambda_{12} \geq x_{21}/(x_{11} + x_{21})$. When this equation is written for x_{22} it takes the form

$$\beta_2^3(\alpha - 1)x_{22}^3 - \beta_2^2(\alpha^2 - 3\alpha + 2)x_{22}^2 + \beta_2(1 - \alpha)(2\alpha - 1)x_{22} - \alpha(2\alpha - 1) \leq 0. \quad (A12)$$

This condition is satisfied if the third order polynomial is negative for $x_{22} \geq 0$. Since $\beta_2^3(\alpha - 1) < 0$ and $-\alpha(2\alpha - 1) < 0$ the polynomial obtains negative values for $x_{22} \geq 0$ or has two positive real roots. Since $-\beta_2^2(\alpha^2 - 3\alpha + 2) \leq 0$ for $0 \leq \alpha \leq 1$ the Descartes ´ rule of signs rules out the possibility of two positive real roots implying that condition (A12) is satisfied. Thus it is shown that there exists a cyclical stationary solution that satisfies all the KT conditions. Finally, under the assumptions applied above equation (31) yields $x_{2\infty} = \sqrt{2}[\sqrt{1 - \alpha} + \sqrt{2}(\alpha - 1)]/[\beta_2(\alpha + 1)]$. In addition: $q = (\sqrt{2}\alpha\sqrt{1 - \alpha} + \alpha - 1)/[\alpha(\alpha + 1)]$. Given $\alpha \in (1/2, 1)$, it follows that in addition to equilibrium cycle there exists an unique interior steady state solution for x_1, x_2 and q. ∎

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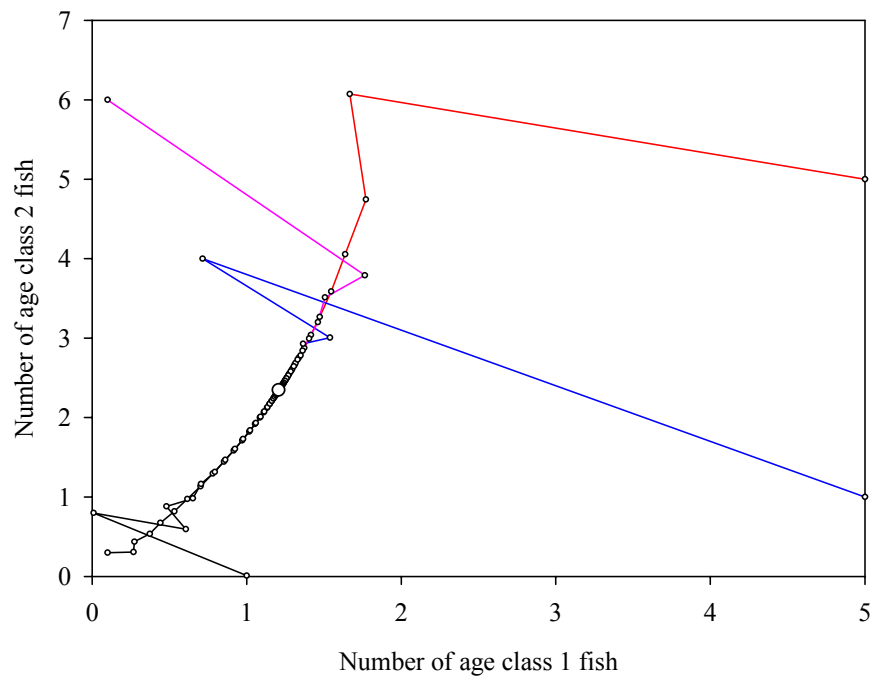
Figure 1. Stability of interior steady state

Parameter values: $r=0.01$, $U=H^{0.5}$, $C=0$, $q_1=0$, $q_2=1$, $\alpha_1=\alpha_2=0.8$,

$\varphi(x_2)=x_2/(1+0.4x_2)$, $\phi_1=0$, $\phi_2=3$
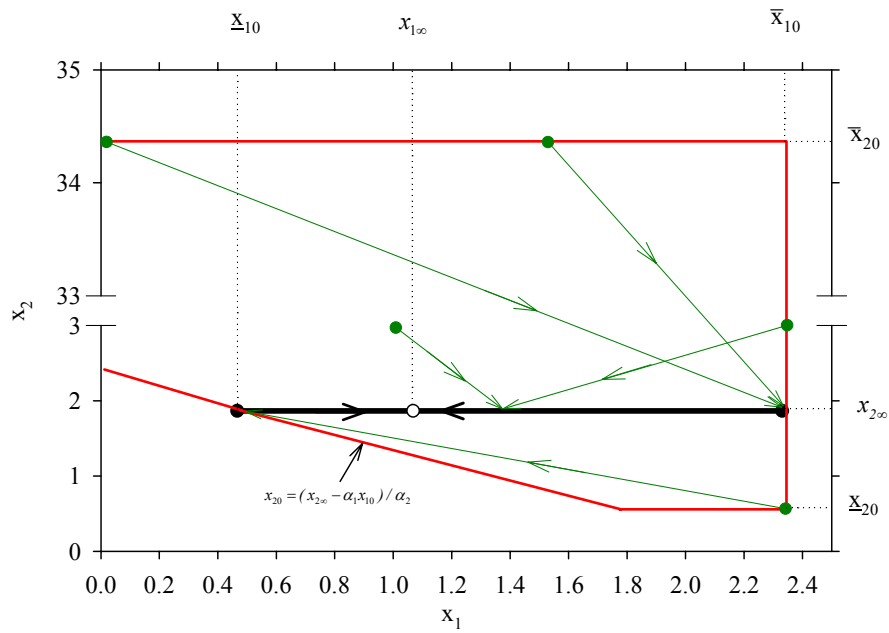
Figure 2a. Local stability of the interior steady state under linear utility

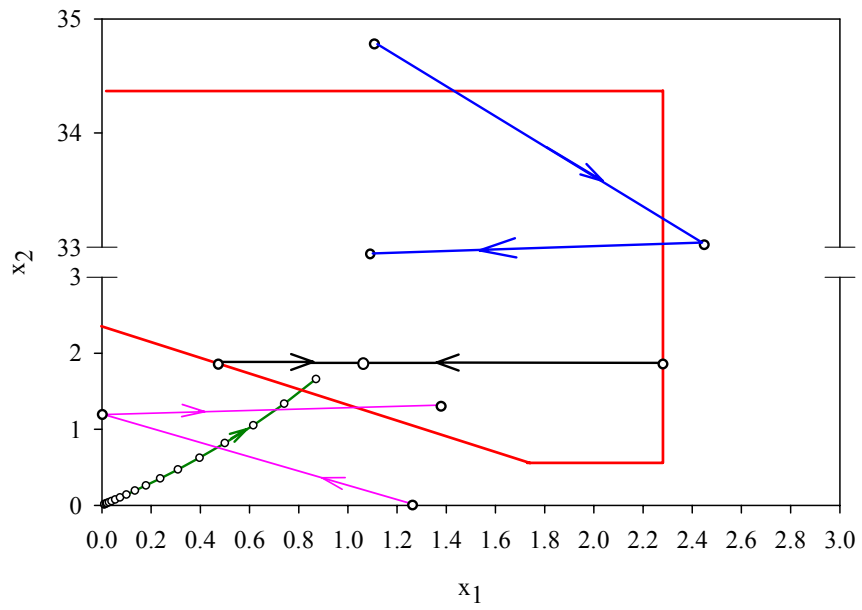Parameter values: r=0.05, $\varphi(x_2)=x_2/(1+0.4x_2)$, $\alpha_1=\alpha_2=0.8$, $\phi_1=0$, $\phi_2=1$, C=0.

Figure 2b. Examples of solutions that reach the steady state within more than two periods
Parameter values: same as in Figure 1a
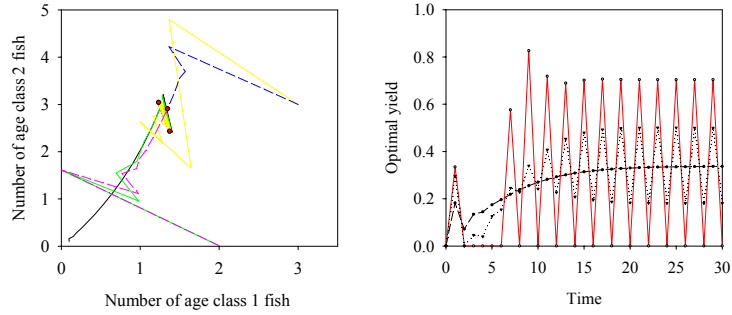
Figure 3a,b. Optimal cycle and the effect of strictly concave utility function

In Figure 3a: solid lines U=H, dashed lines U=$H^{0.5}$

In Figure 3b: solid line U=H, dotted line U=$H^{0.93}$, dashed line U=$H^{0.5}$

In Figures 3a-d: $\varphi(x_2)=x_2/(1+0.4x_2)$, $\alpha_1=\alpha_2=0.8$
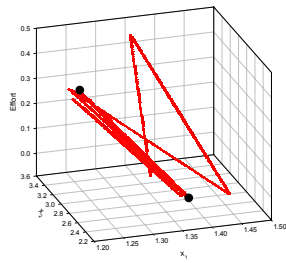
Figure 3c. Optimal cycle in effort,
x_1, x_2 state space.
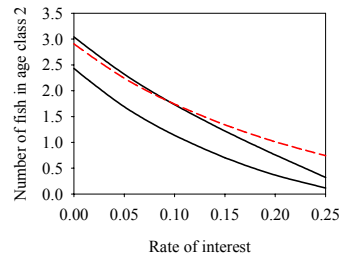
Figure 3d. Optimal steady state and
cycle values for age class 2
as functions of the rate of interest

——— Lower and higher cycle values for x_2

– – – Steady state values
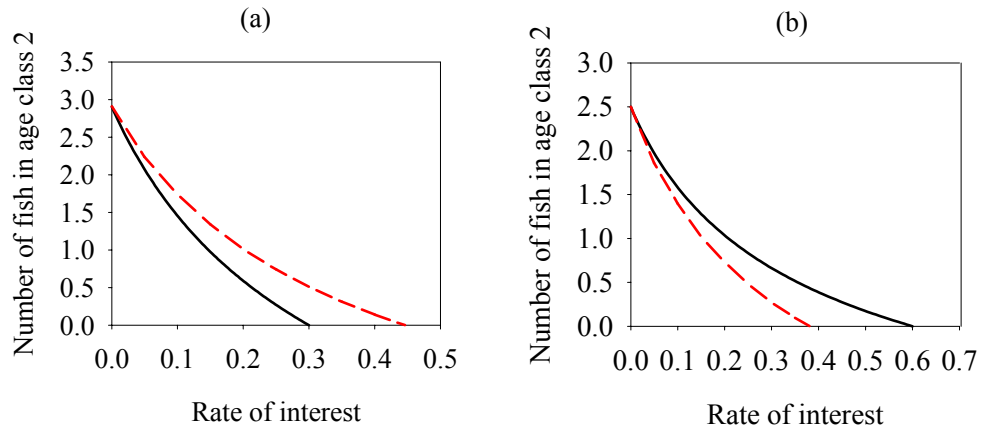
Figure 4. Comparision of steady states of the age-structured and the biomass models
     a) Nonselective gear
     b) Selective gear

Parameters: $\varphi(x_2)=x_2/(1+0.4x_2)$, $\alpha_1=\alpha_2=0.8$, C=0

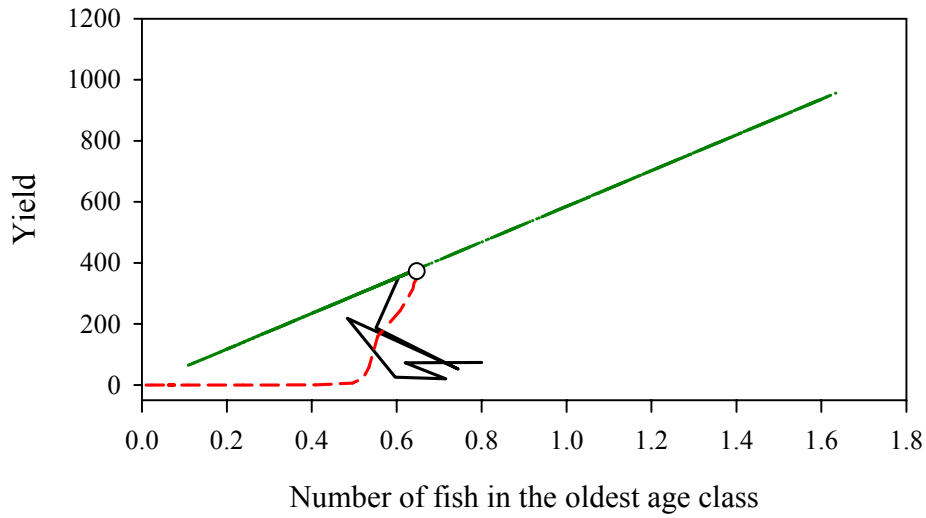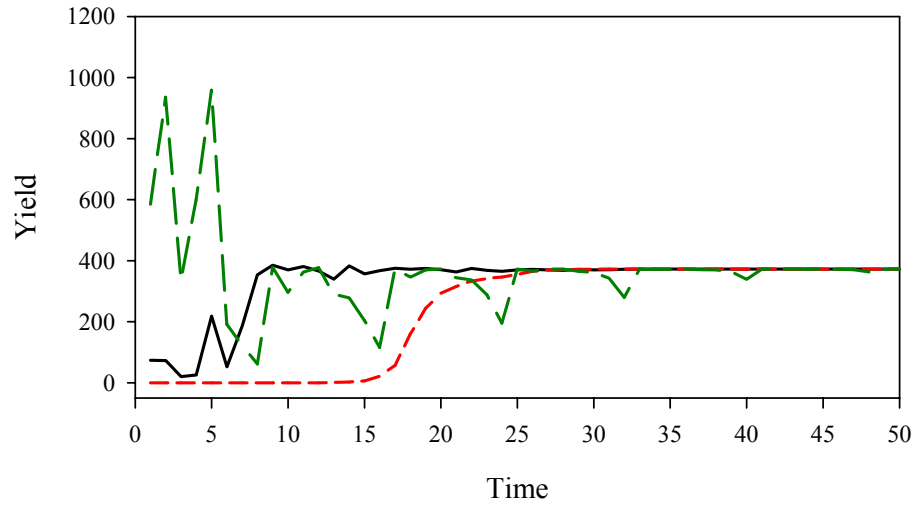———— Biomass model
– – – Age-structured model

Figure 5a,b. Optimal solutions with three initial age distributions
Parameters: $\varphi(x_8)=x_8/(1+0.6x_8)$, $U=H^{0.9}$, $r=0.01$, $q_8=1$, $\alpha=0.8$, $C=0$
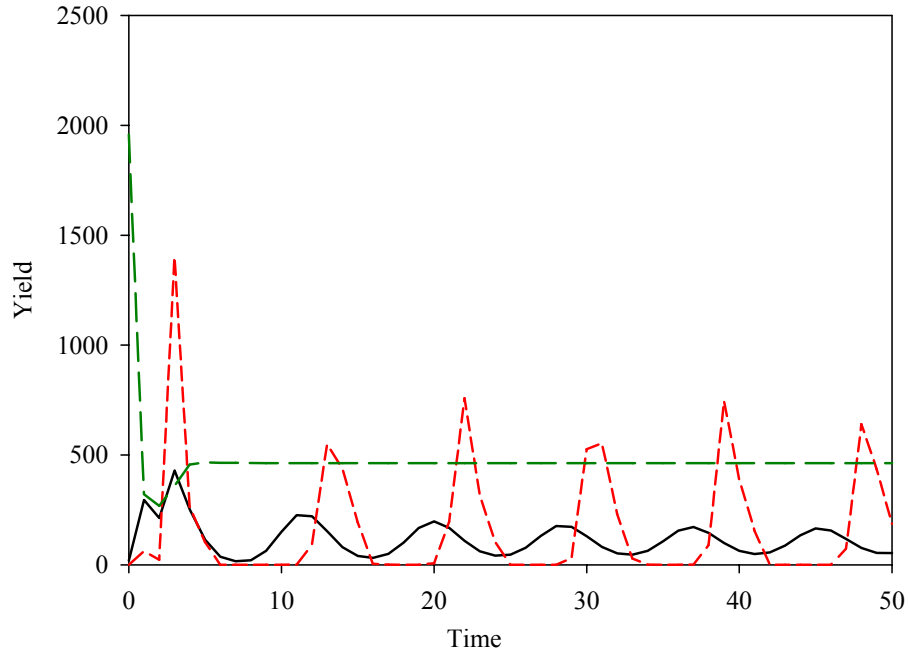$\gamma_1,...,\gamma_7=0$, $\gamma_8=479$, $\phi_1,...,\phi_7,=0$, $\phi_8=731$

Figure 6. Optimal solutions for the general model
Parameters: $U=H^{\sigma}$, $r=0.01$, $\varphi=0.0205x_{0t}e^{-0.0024x_{0t}}$, $\alpha=0.8$

$\sigma=0.5$, $\phi_1,...,\phi_2=0,\phi_5=614,\phi_6=671,\phi_7=707,\phi_8=731$

$\sigma=0.5$, $\phi_1,...,\phi_2=0,\phi_5=614,\phi_6=671,\phi_7=707,\phi_8=731$

$\sigma=0.9$, $\phi_1=100$, $\phi_2=260$, $\phi_3=410$, $\phi_4=530$, $\phi_5=614,\phi_6=671,\phi_7=707,\phi_8=731$