A Numerical Procedure for Simulating Models of Endogenous Growth*

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Abstract

We propose a method to simulate the transition process of endogenous growth models. Different to existing procedures, this method requires the original, unscaled system of differential equations as the only input. It merely has to fulfill sufficient conditions which are satisfied by every existing model of economic growth. Therefore, this algorithm can simulate growth models, for which the analytical calculation of the balanced growth rate is too complicated or even impossible. Internally the algorithm conducts the scaling of the Lucas (1988) model in a general way. By computing the balanced growth rates numerically, the analytical connection to the underlying parameters can be unspecified. In addition, theoretical properties of scale-adjustment are investigated in detail.

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1 Introduction

The analysis of endogenous growth models very often leads to a system of nonlinear differential equations, potentially augmented by static equations such as equilibrium conditions. Usually these equations describe the motion of absolute or per-capita variables which are therefore growing during transition and on a balanced growth path. The first step of analysis normally was to compute the constant balanced growth rates analytically. They are needed not only to analyze the behavior of the model on a balanced growth path, but also to transform the variables into stationary ones, which do not grow in the long run. Usually, analytical and numerical analysis is now conducted with this transformed system.

In many models the analytical connection between the balanced growth rates and the underlying parameters are very complicated (e.g. Romer, 1990, or Lucas, 1988) or even impossible to derive (e.g. endogenous growth computable general equilibrium models, which exhibit a non-linear engine of growth). In this case the above described methods of analysis is laborious or not feasible. If the balanced growth rates cannot be computed analytically, it is impossible to derive conclusions about the balanced growth path or even about the transition towards it, since also the scale-adjusted system of differential equations cannot be obtained.

This paper proposes a method to simulate transitional dynamics of endogenous growth models, for which only the original, unscaled dynamic system is available as an input for the algorithm. Therefore, an analytical calculation of the balanced growth rates is not necessary for this procedure, and the connection between balanced growth rates and the underlying parameters can be unspecified. This is achieved by a numerical computation of the balanced growth rates which are used to conduct the scale-adjustment proposed by Lucas (1988) in a general way. This scale-adjustment always leads to a continuum of saddle-point stable stationary equilibria (i.e. a center manifold). Transitional dynamics of systems exhibiting this characteristic can easily be computed by the relaxation algorithm as proposed by Trimborn, Koch and Steger (2007). Therefore, we make explicitly use of the fact that the relaxation algorithm can solve this kind of problems numerically.

In the context of growth theory, the most prominent approaches to simulate the transition process comprise shooting (e.g. Judd, 1998, Chapter 10), time elimination (Mulligan and Sala-i-Martin, 1991), backward integration (Brunner and Strulik 2002), the projection method (Judd, 1992) as well as the discretization method of Mercenier and Michel (1994 and 2001). A detailed comparison of the similarities and differences of the relaxation procedure can be found in Trimborn, Koch and Steger (2007). Mercenier and Michel (2001) focus on the simulation of endogenous growth models. However, their approach only allows to simulate endogenous growth computable general equilibrium models, which exhibit a linear engine of growth.[1] Relevant in this context is the fact that to our knowledge none of these procedures can simulate the original, unscaled system. In addition, to our knowledge the relaxation algorithm is the only method which can simulate dynamic systems with a center manifold without undergoing fundamental modifications. Therefore, it is the combination of the numerical scale-adjustment and the advantages of the relaxation algorithm which makes the difference to conventional methods.

In detail this paper contributes to two main fields. First of all, the definition of a balanced growth path (BGP) is initially a restriction of the time path of the variables. To exploit these conditions for numerical calculations, we have to transform them into algebraic equations. We derive necessary and sufficient conditions that have to be satisfied along a balanced growth path. These conditions take the shape of equations which can be derived directly from the dynamic system. With these equations we can calculate a point on this path and, moreover, the balanced growth rates (BGR) numerically.

Secondly, we investigate the theoretical properties of the scale-adjustment as presented in Lucas (1988) which slows down the motion of variables by their respective balanced growth rate. By formulating sufficient conditions for the scale-adjustability of dynamic systems, we strengthen the theoretical justification for our approach. Moreover, we proof that the balanced growth path in the phase space is transformed into a center manifold of stationary equilibria.

In summary, the advantage of this algorithm is that the scaling is done numerically, which allows the user to give the original system as the only input. Since sometimes scaled variables are more informative, the algorithm gives both, the original and the scaled variables together with their corresponding growth rates, as an output. As an illustrative application, we apply the procedure to simulate the transition process of the Lucas (1988) model.

In section 2, we present the procedure of automatical scaling of growth models. After reviewing and introducing some mathematical concepts we derive algebraic equations representing the balanced growth path, describe the properties of a numerically scaled system and sketch the structure of the algorithm. In section 3, we evaluate the algorithms' performance on the Lucas (1988) model. Section 4 summarizes and concludes. In the appendix (Section 5) we collect two formal proofs. The extension of the relaxation algorithm has been programmed in MatLab 7.1 and is available on request.

[1] Mercenier and Michel (2001) solve a multi-sector endogenous growth model numerically, which exhibits a linear human capital accumulation equation in the tradition of Lucas (1988).

2 Simulation of endogenous growth models with numerical scaling

To simulate an endogenous growth model with numerical scaling one has to conduct two major steps. The first one is to calculate a point of the balanced growth path and the balanced growth rate from the original, unscaled system. After reviewing some mathematical concepts we derive algebraic equations representing the balanced growth path which are numerically utilizable. The second step is to scale the system numerically. Therefore, we investigate the properties of a numerically scaled system a la Lucas (1988). In the last part of this section we sketch the structure of the algorithm.

2.1 Preliminaries

In this section we want to review some basic definitions and introduce new ones, to set the formal framework for the analysis.

Since our results do not only hold for a system of autonomous differential equations but also for a model which is augmented by algebraic equations we review the definition of a differential algebraic system.[2]

Definition 1 (Differential Algebraic System) *An autonomous Differential Algebraic System (DAS) is a system of which the dynamic behavior is described by a combination of ordinary differential and algebraic equations*

$$\begin{aligned} \dot{X} &= F(X,Y) \\ 0 &= G(X,Y) \end{aligned} \tag{1}$$

The dimension of X *(*n_X*) and* Y *(*n_Y*), match the dimension of* F *and* G*, respectively.*

A balanced growth path is a trajectory of (1) with constant growth rates. To exclude trivial cases we require at least one growth rate to be nonzero. However, we allow the balanced growth rates of variables to be different.

The concept of homogeneity of a function is well known. In the following analysis it will turn out to be of great importance. However, if we would restrict our analysis to homogeneous systems we would automatically require the balanced growth rates of all variables to be the same. To keep the analysis more general we introduce the concept of semi-homogeneity, which includes the homogeneous case. E.g., the Lucas (1988) model will turn out to have a system of differential

[2]We will assume the differential algebraic system to be of differential index one. Systems of higher differential index exhibit far more complex properties and are therefore excluded from analysis (see e.g. Ascher and Petzold, 1998, pp. 231).

equations that is not linear homogenous but only linear semi-homogenous for a positive external effect of human capital in final output production.

Definition 2 (Semi-Homogeneity) *A DAS is semi-homogeneous of degree α with vector $v \epsilon \mathbb{R}^n$, $n_X + n_Y = n$ and one $v(i) \neq 0$ if for any (X, Y) satisfying the algebraic constraint $G(X, Y) = 0$*

$$F(\Lambda_X X, \Lambda_Y Y) = \Lambda_X^{\alpha} F(X, Y) \tag{2}$$

$$G(\Lambda_X X, \Lambda_Y Y) = 0. \tag{3}$$

holds for all $\lambda \epsilon \mathbb{R}^+$ with the matrices $\Lambda_X \epsilon \mathbb{R}^{n_X, n_X}$ and $\Lambda_Y \epsilon \mathbb{R}^{n_Y, n_Y}$ defined as the diagonal matrices of $\lambda^{v(i)}$ for all $i = 1, \ldots, n_X$ and for all $i = n_X + 1, \ldots, n_Y$, respectively. Note that v is not unique but from the linear space $cv, c \epsilon \mathbb{R}$.

In the following, we assume the DAS to be semi-homogeneous on the full domain of definition. This is a sufficient condition for numerical scaling. The question arises how restrictive this condition is. In fact, semi-homogeneity is not a necessary condition for numerical scaling, but it has to be considered that it is necessary to hold on a balanced growth path since there the system has to be semi-homogeneous with vector v representing the vector of balanced growth rates. In addition, to our knowledge it is fulfilled also on the full domain of definition by every existing model which has a balanced growth path. Therefore, we do not consider semi-homogeneity to be a restriction for existing models of economic growth at all.

The last concept we want to present in a general context is the scale adjustment of DAS representing models of economic growth. To our knowledge, this was first introduced by Lucas (1988), who slowed down the motion of variables according to their respective balanced growth rates to conduct analytical investigations. Since it is not naturally given that this modification of the dynamic system will again yield an autonomous DAS we will define the property of simply scale adjustability for DAS.[3]

Definition 3 (Simply Scale Adjustable) *Consider a DAS of the shape (1) and a vector $c \,\epsilon\, \mathbb{R}^n$, $c \neq 0$ such that the transformed system is defined as $x(i) := X(i)e^{-c(i)t}$, $i = 1, \ldots, n_X$, $y(j) := Y(j)e^{-c(j)t}$, $j = n_X + 1, \ldots, n$. If the resulting DAS is autonomous the system is called simply scale adjustable:*

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y) \end{aligned}$$

Below we will present sufficient conditions for simply scale adjustability.

[3] I thank Karl-Josef Koch for helpful comments and fruitful discussions, especially about this topic.

2.2 Expressing the BGP-property by algebraic equations

The property of balanced growth is a condition for the time path of the variables on a subdomain of definition since it constitutes a relation between the absolute value and the value of the first time derivative. The aim is to calculate one point of this trajectory numerically since then the balanced growth rates can be calculated and, therefore, a numerical scaling can be conducted. Note, that under the assumption of semi-homogeneity once a DAS is in balanced growth it will exhibit it forever. Therefore, the whole trajectory is established by the defining equation of semi-homogeneity. However, the defining equation of balanced growth alone cannot be exploited to calculate a point of this path, since the DAS contains the information about the properties of the time-path only implicitly. Therefore we want to construct equations which define the balanced growth path in the phase space unambiguously. Under the assumption of semi-homogeneity, these equations will be necessary and sufficient.

To be more formal, consider a DAS which is defined on $\mathbb{R}^n$. Let us further assume that the balanced growth path is an invariant manifold of dimension n_b. That is, we are looking for $n - n_b$ equations in $\mathbb{R}^n$ which define the BGP manifold.

Definition 4 (BGP equations) *Consider a DAS of shape (1) with n_X differential equations and n_Y static equations such that $n_X + n_Y = n$. We define the associated BGP equations as*

$$D_X F \cdot F(X,Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X,Y) - \Gamma_X(X,Y) \cdot F(X,Y) = 0 \tag{4}$$

together with the static equations. The matrix $\Gamma_X(X,Y)$ is defined as the diagonal matrix of $\frac{F_i(X,Y)}{X_i}$, $i = 1 \dots n_X$. If static equations are absent, the equations reduces to

$$D_X F(X) \cdot F(X) = \Gamma(X) \cdot F(X). \tag{5}$$

Theorem 5 (Necessary and sufficient conditions for BGP) *Consider a DAS of shape (1) with n_X differential equations and n_Y algebraic equations such that $n_X + n_Y = n$. Then a vector X exhibits the BGP property if and only if the associated BGP equations hold. Note, that equations (4) and (5), respectively are a system of n_X equations whereas only $n_X - n_b$ are linear independent.*

Moreover, if the DAS is semi-homogeneous with vector $v \epsilon \mathbb{R}^n$ where $v_1, \dots, v_{n_X}$ represents the vector of balanced growth rates of X, then Y also is on a balanced growth path with growth rates $v_{n_X+1}, \dots, v_n$.

Proof. Differentiating the static equation with respect to time yields

$$\begin{aligned} & D_Y G \cdot \dot{Y} + D_X G \cdot F(X,Y) = 0 \\ \Leftrightarrow \quad & \dot{Y} = -D_Y G^{-1} \cdot D_X G \cdot F(X,Y) \end{aligned} \tag{6}$$

We assume, that the inverse exists.[4] Now, differentiating the equation $F(X,Y) = \Gamma_X(X,Y) \cdot X$ with respect to time and applying the BGP property yields

$$D_X F \cdot F(X,Y) + D_Y F \cdot \dot{Y} = \Gamma_X(X,Y) \cdot F(X,Y)$$

Substituting for $\dot{Y}$ yields the n_X dimensional equation

$$D_X F \cdot F(X,Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X,Y) - \Gamma_X(X,Y) \cdot F(X,Y) = 0$$

These equations do not require the static equations to hold. Therefore, together with the n_Y static equations the BGP manifold is well defined.

To proof sufficiency of the equations for balanced growth assume that the balanced growth assumption for X does not hold necessarily. Reconsidering the calculations above the equations have to be augmented by an additional term embodying the change of the growth rates with respect to time.

$$\begin{aligned} & D_X F \cdot F(X,Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X,Y) \\ & -\Gamma_X(X,Y) \cdot F(X,Y) - \frac{d\Gamma_X(X,Y)}{dt} X = 0 \end{aligned}$$

The last summand is zero if and only if X is on a BGP.

It remains to proof that if X is on a BGP and the system is semi-homogeneous, Y will also fulfill the BGP property. Semi-homogeneity was not exploited until now.

Differentiating the algebraic equation of semi-homogeneity with respect to λ and evaluating at $\lambda = 1$ and for an (X,Y) with $G(X,Y) = 0$ yields

$$D_X G(X,Y) diag(v_X) X + D_Y G(X,Y) diag(v_Y) Y = 0$$

whereby $diag(v_X)$ and $diag(v_Y)$ are diagonal matrices with X and Y components of the vector v of semi-homogeneity in the diagonal. Subtracting equation (6) yields

$$D_X G(X,Y)(diag(v_X) X - F(X,Y) X) + D_Y G(X,Y)(diag(v_Y) Y - \dot{Y}) = 0$$

[4]This is a sufficient condition for the DAS to be of differential index one. It is not necessary, however, it is only slightly stronger.

Since the first bracket is zero and $D_Y G$ is regular, it follows directly that Y is in balanced growth with the growth rates being equal to the vector of semi-homogeneity, because balanced growth rates and the vector of semi-homogeneity are the same for the dynamic variables. ■

These equations can be exploited to calculate the balanced growth path numerically, using a conventional method to calculate the roots. We want to clarify that these equations would also yield an analytical formulation of the balanced growth path.

To make the properties of the equations clear we present a concise example. Consider the Ramsey-Cass-Koopmans model (see Ramsey, 1928, Cass, 1965 and Koopmans, 1965) without technological progress. The model's behavior is represented by the following set of differential equations (see Barro and Sala-i-Martin, 2004, Chapter 2).[5]

$$\dot{C} = \frac{C}{\theta}\left(\alpha K^{\alpha-1} L^{1-\alpha} - (\delta+\rho)\right) + nC \quad (7)$$

$$\dot{K} = K^{\alpha} L^{1-\alpha} - C - \delta K \quad (8)$$

$$\dot{L} = nL, \quad (9)$$

where α denotes the elasticity of capital in production, n the population growth rate, δ the depreciation rate, ρ the parameter for time preference and θ the inverse of the intertemporal elasticity of substitution, respectively. Evaluating equation (5) for this model and simplifying yields the equations $K = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} L$ and $\frac{C}{L} = \left(\frac{K}{L}\right)^{\alpha} - (n+\delta)\frac{K}{L}$.[6] Note that these are only two independent relations from originally three equations which indicates the dimension one of the balanced growth path. In this comparatively simple model these equations can be simplified such they give a good analytical representation of the balanced growth path. However, in more complicated models potentially these equations can only be exploited numerically.

For some models, we experienced that finding the roots of equations (4) and (5), respectively, is an ill-conditioned problem.[7] Therefore, we want to exploit the property of semi-homogeneity to construct a different approach for calculating a point of the balanced growth path.[8] The approach is to calculate the vector of semi-homogeneity numerically in a first step. In a second step, this information is exploited to construct a different set of non-linear equation, which solution again represents the balanced growth path. The solution of this set of equation has turn out to be well conditioned.

[5] Usually, the model is presented in per capita notation. However, here this would be a representation in non growing variables. Therefore we choose the formulation in absolute units which are denoted in capital letters.

[6] This, of course, would be tedious without using computer algebraic programs such as MatLab or Mathematica which can conduct analytical calculations.

[7] Cite press et al about roots of non-linear equations

[8] I would like to thank Jan Schlemmer for fruitful discussions about this topic.

Corollary 6 (Vector of semi-homogeneity) *Consider a DAS that is semi-homogenous with vector v. Then, for any (X, Y) satisfying the algebraic constraint $G(X, Y) = 0$, v is in the kernel of the matrix*

$$M := \begin{pmatrix} D_X F & D_Y F \\ D_X G & D_Y G \end{pmatrix} diag \begin{pmatrix} X \\ Y \end{pmatrix} - diag \begin{pmatrix} F(X, Y) \\ 0 \end{pmatrix}$$

Proof. We differentiate the defining equation of semi-homogeneity with respect to λ and evaluate for $\lambda = 1$ to get

$$\begin{aligned} D_X F(X,Y) diag(X) v_X + D_Y F(X,Y) diag(Y) v_Y &= diag(F(X,Y)) v_X \\ D_X G(X,Y) diag(X) v_X + D_Y G(X,Y) diag(Y) v_Y &= 0 \end{aligned}$$

v denotes the vector of semi-homogeneity, and v_X, v_Y its X and Y components, respectively. $diag(\cdot)$ denotes the respective diagonal matrix.

Both equations can be combined to

$$\begin{pmatrix} D_X F & D_Y F \\ D_X G & D_Y G \end{pmatrix} diag \begin{pmatrix} X \\ Y \end{pmatrix} v - diag \begin{pmatrix} F(X, Y) \\ 0 \end{pmatrix} v = 0 \tag{10}$$

■

Note, that $Mv = 0$ is an $n = n_X + n_Y$ dimensional linear equation. Since the system is semi-homogenous, the vector v is from the same linear space, wherever (10) is evaluated.

Now, we construct a set of non-linear equations, which solution is a balanced growth path. This set of equations exploits the information about the vector of semi-homogeneity.

Theorem 7 (Balanced growth of systems exhibiting semi-homogeneity) *Consider a DAS that is semi-homogenous with vector v, and a point (X, Y) that satisfies $G(X, Y) = 0$. We assume the system to exhibit a one-dimensional BGP. Then, if and only if X is in balanced growth it satisfies*

$$\frac{F_i(X,Y)}{X_i} \frac{1}{v_i} = \frac{F_j(X,Y)}{X_j} \frac{1}{v_j} \qquad \forall \quad 1 \leq i, j \leq n_X \tag{11}$$

Moreover, if X is in balanced growth this also follows for Y.

Proof. The proof is relegated to the appendix. ■

Equation (11) together with $G(X, Y) = 0$ represents a system of $n-1$ equations for n unknowns. This indicates that we assumed the BGP to be of dimension one. Once a point of the balanced growth path is calculated the balanced growth rates can be calculated easily by evaluating F and G at this point.

2.3 Numerical scale-adjustment of growth models

The next step is to scale the model in a general way such that all variables approach a constant in the long run. Since this scaling has to be done in a general way we want to employ scale adjustment as presented in Lucas (1988) to scale the model numerically. Scale adjustment slows down the motion of variables according to their respective balanced growth rates. Here, the calculation of these growth rates is done numerically. Therefore, we have to derive the shape of the scaled system in a general form. It is not naturally given that this scale adjustment will finally result in an autonomous system. However, we can prove that under the assumption of semi-homogeneity this will always be the case.

Theorem 8 (Sufficient criteria for scale-adjustability) *If a DAS is linear semi-homogeneous with vector v, then the scale-adjusted version is autonomous with growth rates $v(i)$, and the scaled system reads*

$$\dot{x} = F(x, y) - Vx \quad (12)$$

$$0 = G(x, y) \quad (13)$$

with $V \epsilon \mathbb{R}^{n_X, n_X}$ representing the diagonal matrix of v.

Proof. From the definition $x = \Lambda_X X$, $y = \Lambda_Y Y$ with Λ_X, Λ_Y representing diagonal matrices of $e^{-v(i)t}$, $i = 1, \ldots, n_X$ and $e^{-v(j)t}$, $j = n_{X+1}, \ldots, n$, respectively, it follows that:

$$\begin{aligned} \dot{x} &= -V\Lambda_X X + \Lambda_X \dot{X} \\ &= -V\Lambda_X X + \Lambda_X F(X, Y) \\ &= -V\Lambda_X X + F(\Lambda_X X, \Lambda_Y Y) \\ &= -Vx + F(x, y) \end{aligned}$$

where V denotes the diagonal matrix of v. Analogous for G. ■

Scale adjustment transforms a specific balanced growth path into a stationary point. In addition, we have to investigate the properties of the scaled system to verify that the scaled system can be solved numerically. It turns out that the balanced growth path trajectory is transformed into a center manifold of stationary points.[9]

[9] For details on the concept of center manifolds see, for instance, Tu (1994).

Theorem 9 (Properties of DAS with Balanced Growth Paths)
Consider a DAS of shape (1) that possesses a BGP solution which we will denote with an asterisk. Let us assume that there exists an autonomous, normalized version of (1) of the shape

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y) \end{aligned} \tag{14}$$

with $x(i) := X(i)e^{-c_1(i)t}$, $y(j) := Y(j)e^{-c_2(j)t}$ *for all components, whereby* c_1, c_2 *are the constant balanced growth rates. It follows that system (14) has a fixed point in at least one point of time for a BGP solution:*

$$\begin{aligned} 0 &= f(x^*, y^*) \\ 0 &= g(x^*, y^*) \end{aligned}$$

Then x^* *and* y^* *lie on a center manifold which moreover consists of stationary points. This center manifold has the same shape in the* (x, y)*-space of (14) as an invariant manifold in the* (X, Y)*-space of (1), satisfying the BGP property.*

Proof. The proof is relegated to the appendix. ■

The fact, that the numerically scale adjusted system exhibits a center manifold of stationary points demands a qualitative investigation of the dynamics in the neighborhood of a stationary point. It is not possible to apply the theorem of Hartman and Grobman, since it explicitly excludes the analysis of non-hyperbolic points.[10] However, it is possible to apply the fundamental theorem of normally hyperbolic manifolds. (For a statement of the theorem see Hirsch, Pugh and Shub, 1977, or Li et al., 2003) The center manifold is normally hyperbolic, since it exhibits an eigenvalue of zero with a corresponding eigenspace tangent to the center manifold. Let us assume that apart from n_c eigenvalues of zero associated to a n_c dimensional center manifold (CM) of stationary points there exists n_s stable and n_u unstable eigenvalues.[11] For local stability analysis we want to recall the following conclusions from the theorem.

- There exits a local unstable and stable manifold of the center manifold, W^u and W^s, of dimension n_u+n_c, n_s+n_c, respectively, tangent to the subspace spanned by the corresponding eigenvectors. This means, a trajectory converging to the center manifold is locally unique if and only if $n_s + n_c$ initial conditions are given.

[10] For a statement of the theorem see for instance Tu (1994, p. 135).
[11] A stable or unstable eigenvalue in this context is an eigenvalue with a real part smaller or greater than zero, respectively.

- Foliation: W^u and W^s are fibred by submanifolds (leaves) W_x^{uu}, W_x^{ss}, $x \epsilon CM$ of dimension n_u and n_s, respectively, tangent to the subspace spanned by the corresponding eigenvectors. W_x^{uu} and W_x^{ss} are characterized such that all points of W_x^{uu} and W_x^{ss} converge exponentially to x as $t \to -\infty$, $t \to \infty$, respectively. This means, all economies represented by W_x^{ss} form a class, which converges to the same stationary point in the long run. The direction of convergence is the subspace, generated by the stable eigenvectors. Since there is no movement on the CM the qualitative behavior is the same as for the linearized system.

The first point indicates that with this scale adjustment it can be easily analyzed wether a specific optimal control problem yields a unique trajectory. Moreover, the second point indicates that this proceeding allows for a distinction of different balanced growth paths and therefore more information is provided.

In general it is not possible to determine analytically, to which stationary equilibrium the economy converges. The relaxation algorithm is able to solve this kind of problem numerically. Since final boundary conditions can be given in terms of equations representing the center manifold the algorithm does not need the information about the particular stationary equilibrium to reach in advance.

To clarify the question of dimensionality consider the model as described in Lucas (1988). The scale adjusted model has a one dimensional center manifold of stationary points. Since the phase space is four-dimensional and the model consists of two state variables, a trajectory is locally unique if and only if the Jacobian matrix evaluated at the center manifold exhibits one stable and two unstable eigenvalues. The forth eigenvalue is zero and associated with the center manifold. This is the case for the set of parameters used in Lucas (1988) or Caballe and Santos (1993).[12]

2.4 Design of the algorithm

The algorithm is programmed in MatLab 7.1 and available on request. It builds on the version of the relaxation algorithm proposed by Trimborn et al. (2007).[13] In the first step, the algorithm solves for a point of the balanced growth path and calculates the balanced growth rates whereas in the second step the traditional relaxation algorithm is applied to solve for transitional dynamics of the numerically scaled DAS. In detail the algorithm proceeds as follows.

- Calculation of an admissible (X, Y)

  A vector (X, Y) has to be calculated, which satisfies the algebraic constraint $G(X, Y) = 0$.

[12]However, this is not the case for all reasonable sets of parameters as shown in Benhabib and Perli (1994).

[13]For a more general description of relaxation see, for instance, Press et al. (1989).

It is either possible to select X and calculate Y by employing the implicit function theorem, or to select an initial guess (X, Y) and minimize the residuum $G(X, Y)$ with an appropriate standard routine. We follow the latter and employ the *fsolve* Matlab procedure, which can handle non-square systems.

- Calculation of the vector of semi-homogeneity

  Calculate the Jacobian matrices DF and DG for an arbitrary (X, Y), and calculate the kernel of M (Solve equation (10)).

- Calculation of BGP

  For constructing BGP-equations, we exploit the representation in equation (19) of the appendix. In this equation, an additional unknown $s \epsilon \mathbb{R}$ is introduced. Therefore, equation (19) together with the algebraic equations exhibits $n_X + n_Y + 1$ unknowns and $n_X + n_Y$ equations. The system can be solved by employing an adequate standard routine, or by converting it to a square system by assuming $X(i) = 1$ for one i with $v(i) \neq 0$.

- Calculation of BGR

  At this single point of the balanced growth path the balanced growth rates are calculated by evaluating $F(X^*, Y^*)$. Thereby, the growth rates of the static variables can be calculated by employing equation (6).[14]

- Numerical Scaling and calculation of transitional dynamics

  Since now the balanced growth rates are known, the system can be scaled numerically and the usual relaxation procedure can be applied. From the solution, the original variables can be recovered by multiplying with their respective balanced growth rate. Just as well the growth rate of each variable dependent on time can be computed.

The crucial point for the algorithm to work is the solution of non-linear equations. This problem can in principle be ill-conditioned without any chance of cure. The solution of non-linear equations enters the algorithm at two stages, which is for the calculation of an admissible (X, Y) and at the solution of the equations given by theorem 7. We did not experience any problems for the models we tested.

[14] It is not necessary to compute the balanced growth rates of Y for the solution of transitional dynamics, but for the interpretation of the simulation results in scale adjusted variables.

3 A concise example: The Lucas model

As an example we simulate the Lucas (1988) model which is also discussed in Mulligan and Sala-i-Martin (1993) and Benhabib and Perli (1994). This model exhibits the interesting characteristic that without an external effect in production the growth rates of output, physical capital and human capital are the same. Introducing an external effect causes the growth rate of human capital to deviate from the other variables' growth rates.

Final output is produced from physical capital k and human capital h. The share u of human capital is used for final output production while the remainder $1-u$ is employed to increase human capital. Due to human capital spill over effects there are increasing returns to scale in the production sector. A representative household maximizes intertemporal utility of consumption c with constant elasticity of intertemporal substitution σ^{-1} and discount rate ρ. The first order conditions for optimal solutions in terms of a system of differential equations read

$$\begin{aligned} \dot{k} &= A\,k^{\alpha}h^{1-\alpha+\gamma}u^{1-\alpha} - c \qquad (15) \\ \dot{h} &= \delta(1-u)h \qquad (16) \\ \dot{c} &= \sigma^{-1}c(\alpha A\,k^{\alpha-1}h^{1-\alpha+\gamma}u^{1-\alpha} - \rho) \qquad (17) \\ \dot{u} &= u\left(\frac{(\gamma-\alpha)\delta}{\alpha}(1-u) + \frac{\delta}{\alpha} - \frac{c}{k}\right) . \qquad (18) \end{aligned}$$

Balanced growth requires u, c/k as well as $k^{\alpha-1}h^{1-\alpha+\gamma}u^{1-\alpha}$ to be constant. The latter requirement in turn demands $(1-\alpha)\frac{\dot{k}}{k} = (1-\alpha+\gamma)\frac{\dot{h}}{h}$.

In this simple case the common balanced growth rate μ of k and c can be computed by solving the system under balanced growth assumptions:

$$\mu = \frac{1-\alpha+\gamma}{(1-\alpha+\gamma)\sigma-\gamma}\,(\delta-\rho)$$

Growth is balanced if the four variables of the system satisfy three equations:

$$\begin{aligned} 1-u &= \frac{1-\alpha}{(1-\alpha+\gamma)\sigma-\gamma}(1-\rho/\delta) \\ c/k &= ((\gamma-\alpha)\psi\mu+\delta)/\alpha \\ k^{\alpha-1}h^{1-\alpha+\gamma} &= \frac{\sigma\mu+\rho}{\alpha A}u^{\alpha-1} \end{aligned}$$

where $\psi := (1-\alpha)/(1-\alpha+\gamma)$.

For numerical computation we pretend not to know the balanced growth rates nor the three equations representing the balanced growth path. The only input for the algorithm is the dynamic system (15-18) together with a set of common parameter values.[15] We calculate the vector of semi-homogeneity v numerically with an arbitrary Y. System (15-18) does not exhibit an algebraic constraint, which is why we can skip the calculation of an admissible vector. Then, we solve for the BGP with equation (19) together with the algebraic equations. For this, we employ the Matlab procedure *fsolve*, which can handle non-square systems.

We can now compare the results of the numerically calculated balanced growth rates with their analytical correspondent (see table 1). The comparison indicates a high degree of accuracy in both, absolute and relative error.[16]

Table 1: Comparison of analytical and numerical calculated balanced growth rates

Variable	BGR (analytical)	BGR (numerical)	relative error
h	0.0291666667	0.0291666670	$< 1.2 \cdot 10^{-8}$
k	0.0416666667	0.0416666659	$< 1.9 \cdot 10^{-8}$
c	0.0416666667	0.0416666665	$< 3.7 \cdot 10^{-9}$
u	0	$-1.2 \cdot 10^{-16}$	∞

In addition, we want to investigate how these small errors propagate in the calculation of the transition path. Therefore we also simulate the analytically scaled model with the relaxation algorithm everything else being equal, assuming that this is a sufficiently precise representation of the exact transition path. The comparison indicate a low propagation of the error for the scaled variables (see Figure 1). The maximum relative error is below $2 \cdot 10^{-7}$ for all variables.[17]

Overall this example suggests that the calculation of the balanced growth rates is sufficiently exact to conduct the scaling of the system numerically.

4 Conclusion

We propose an extension of the relaxation algorithm introduced by Trimborn, Koch and Steger (2007) for simulating the transition process of endogenous growth models. The main advantage of this method is that it requires the original, unscaled system of differential equations potentially augmented by algebraic equations as the only input. Therefore, this algorithm can simulate growth models where the analytical calculation of the balanced growth rate is too complicated or

[15] The set of chosen parameters implies a determinate adjustment path, see Benhabib and Perli (1994). They are $A = 1$, $\beta = 0.3$, $\gamma = 0.3$, $\delta = 0.1$, $\rho = 0.05$ and $\sigma = 1.5$.

[16] An exception, of course, is the relative error of the growth rate of u, since the true value is zero.

[17] As a shock, we assume the initial value of human capital h to be 50% above its steady state value.

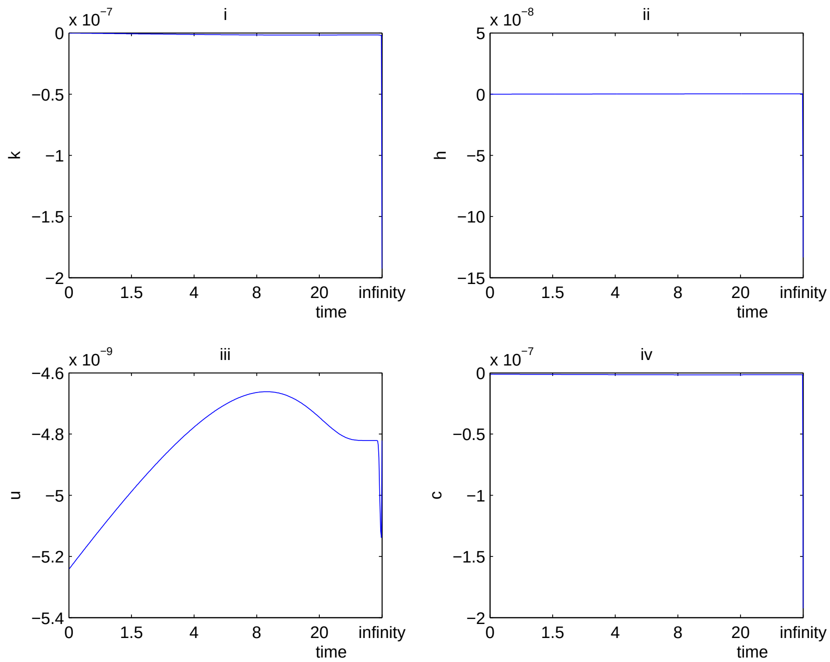

Figure 1: Relative error at the (nonlinear) time scale

even impossible. We derived algebraic equations representing the balanced growth path to exploit the balanced growth path property numerically. Furthermore we investigated the properties of a dynamic system which has been numerically scale adjusted similarly as first proposed by Lucas (1988).

Although the goal has been to ease the numerical treatment of endogenous growth models, the developed methods could also be used on various other issues. Both, the algebraic equations representing a balanced growth path and the properties of a scale adjusted model can also be exploited analytically and are therefore potentially of additional value.

5 Appendix

5.1 Proof of theorem 7

Proof. The claim can be restated as

$$F(X,Y) = s(X,Y)diag(v_X)X, \quad s:(\mathbb{R}^{n_X},\mathbb{R}^{n_Y}) \rightarrow \mathbb{R} \qquad \Leftrightarrow \qquad X \,\epsilon\, BGP \tag{19}$$

We assume X to exhibit balanced growth. From the defining equation of semi-homogeneity it follows by differentiating with respect to λ and evaluating for $\lambda = 1$ that

$$\begin{aligned} D_X F(X,Y) diag(X) v_X + D_Y F(X,Y) diag(Y) v_Y &= diag(F(X,Y)) v_X \\ D_X G(X,Y) diag(X) v_X + D_Y G(X,Y) diag(Y) v_Y &= 0 \end{aligned}$$

for an (X,Y) that satisfies $G(X,Y) = 0$.

The second equation inserted into the first one yields

$$\begin{aligned} & D_X F(X,Y) diag(v_X) X + D_Y F(X,Y) D_Y G(X,Y)^{-1} D_X G(X,Y) diag(v_X) X \\ & = diag(v_X) F(X,Y) \end{aligned} \tag{20}$$

From the BGP equation is follows that $F(X,Y)$ is in the kernel of

$$D_X F(X,Y) + D_Y F(X,Y) D_Y G(X,Y)^{-1} D_X G(X,Y) - \Gamma_X(X,Y)$$

Rewriting equation (20) yields

$$\left(D_X F(X,Y) + D_Y F(X,Y) D_Y G(X,Y)^{-1} D_X G(X,Y) - \Gamma_X(X,Y) \right) diag(v_X) X = 0$$

Therefore

$$F(X,Y) = s \cdot diag(v_X) X \qquad \Leftrightarrow \qquad \Gamma_X = s \cdot diag(v_X)$$

with some $s \in \mathbb{R}$, if the BGP is one-dimensional.

Now assume the lhs of (19) holds. Differentiating the equation with respect to time and employing the algebraic equation yields

$$\begin{aligned} & D_X F(X,Y) F(X,Y) + D_Y F(X,Y) D_Y G(X,Y)^{-1} D_X G(X,Y) F(X,Y) \\ & = s(X,Y) diag(v_X) F(X,Y) + \frac{ds(X,Y)}{dt} \end{aligned} \tag{21}$$

Multiplying (20) with $s(X,Y)$ and substituting the assumption yields

$$\begin{aligned} & D_X F(X,Y) F(X,Y) + D_Y F(X,Y) D_Y G(X,Y)^{-1} D_X G(X,Y) F(X,Y) \\ & = s(X,Y) diag(v_X) F(X,Y) \end{aligned} \tag{22}$$

Subtracting the equation from (21) yields

$$\frac{ds(X,Y)}{dt} = 0 \tag{23}$$

Hence, X is in balanced growth. ■

5.2 Proof of theorem 9

Proof. Note first, that if a BGP of (1) is a stationary point of (14) in one point of time, then this holds for the maximum interval of definition.[18] Therefore let us assume this holds for time zero, i.e. the positive orbit of the flow ϕ, $\{\phi_i(X_0^*, Y_0^*)\}_{i\epsilon[0,\infty)}$ is a BGP and the corresponding vector (x_0^*, y_0^*) is a stationary point. Note that the latter will be constant for all t. Now consider the positive orbit of the flow ϕ, $\{\phi_i(\phi_t(X_0^*, Y_0^*))\}_{i\epsilon[0,\infty)}$ for some $t > 0$ which lies on the same invariant manifold and therefore exhibits the same BGP property and the same growth rates. This will also define a stationary point (x_1^*, y_1^*), since the limit of the ratio is constant:

$$\lim_{t\to\infty} \frac{X_0^*(i)e^{c_1(i)t}}{X_1^*(i)e^{c_1(i)t}} = \frac{X_0(i)}{X_1(i)} = \lim_{t\to\infty} \frac{x_0^*(i)}{x_1^*(i)} \qquad \forall i,\ 1 \leq i \leq n_X$$

But $(x_0^*, y_0^*) \neq (x_1^*, y_1^*)$ because for time zero $(X, Y) = (x, y)$ and $\phi_0(X_0^*, Y_0^*) \neq \phi_0(\phi_t(X_0^*, Y_0^*))$ since (X, Y) is growing in at least one component. Since the flow is continuous and differentiable in t, this mapping in (x, y) will define a manifold of stationary points which in turn implies the existence of a center manifold. Furthermore, since the mapping $(X, Y) \to (x, y)$ is the identity for $t = 0$ the shape of this center manifold will be the same as that of the invariant manifold. ■

[18] For convenience, the upper case variables will now always refer to (1) and the lower case variables to (14).

References

Ascher, U. M., and L. R. Petzold (1998): *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. Philadelphia : Society for Industrial and Applied Mathematics.

Barro, R. J., and X. Sala-i-Martín (2004): *Economic Growth*. MIT-Press, Cambridge, Massachusetts.

Benhabib, J., and R. Perli (1994): "Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth," *Journal of Economic Theory*, 63, 113–142.

Brunner, M., and H. Strulik (2002): "Solution of Perfect Foresight Saddlepoint Problems: A Simple Method and Applications," *Journal of Economic Dynamics and Control*, 26, 737–753.

Caballe, J., and M. S. Santos (1993): "On Endogenous Growth with Physical and Human Capital," *The Journal of Political Economy*, 101(6), 1042–1067.

Cass, D. (1965): "Optimum Growth in an Aggregate Model of Capital Accumulation," *Review of Economic Studies*, 32, 233–240.

Hirsch, M. W., C. C. Pugh, and M. Shub (1977): *Invariant Manifolds*. Springer-Verlag, New York.

Judd, K. L. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, 410–452.

——— (1998): *Numerical Methods in Economics*. MIT Press.

Koopmans, T. C. (1965): "On the Concept of Optimal Economic Growth," In: *The Econometric Approach to Development Planning,* Amsterdam: North Holland.

Li, W., M. Rychlik, F. Szidarovsky, and C. Chiarella (2003): "On the Attractivity of a Class of Homogeneous Dynamic Economic Systems," *Nonlinear Analysis*, 52, 1617–1636.

Lucas, R. E. Jr.. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3–42.

Mercenier, J., and P. Michel (1994): "Discrete-Time Finite Horizon Approximation of Infinite Horizon Optimization Problems with Steady-State Invariance," *Econometrica*, 62(3), 635–656.

——— (2001): "Temporal Aggregation in a Multi-Sector Economy with Endogenous Growth," *Journal of Economic Dynamics and Control*, 25, 1179–1191.

Mulligan, C., and X. Sala-i-Martin (1991): "A Note on the Time Elimination Method for Solving Recursive Growth Models," *Technical Working Paper No. 116.*

——— (1993): "Transitional Dynamics in Two-Sector Models of Endogenous Growth," *Quarterly Journal of Economics*, 108(3), 737–773.

Press, W., B. Flannery, S. Teukolsky, and W. Vetterling (1989): *Numerical Recipes in Pascal.* Cambridge University Press.

Ramsey, F. (1928): "A Mathematical Theory of Saving," *Economic Journal*, 38, 543–559.

Romer, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

Trimborn, T., K.-J. Koch, and T. M. Steger (2007): "Multi-Dimensional Transitional Dynamics: A Simple Numerical Procedure," *Macroeconomic Dynamics, forthcoming.*

Tu, P. N. (1994): *Dynamical Systems, An Introduction with Applications in Economics and Biology.* Springer-Verlag, Berlin.